Bow and arrow dynamics

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Past analyses of bow and arrow dynamics have assumed the string to be inextensible. This results in predictions of efficiencies that are significantly higher than measured values (efficiencies over 90% are predicted versus 70% to 85% for measurements, circa 1960). The present analysis allows for an elastic string. It is found that arrow exit then takes place when the string and bow limbs still have substantial kinetic energy, and therefore this energy is unavailable for kinetic energy of the arrow. Moreover, the potential energy remaining in the string and bow limb system can also reduce the amount of energy available for the arrow. For the Hickman model of a long bow used in this study, the elastic string prediction of efficiency is 78%, whereas the inelastic prediction is 92%. The analysis utilizes a Lagrangian distributed mass formulation to develop the governing equations of motion and to generate an equivalent point mass model. The equations of motion were numerically integrated to obtain efficiency, arrow velocity, virtual masses, string tension, string extension, arrow exit time, string and limb potential energies, system momentum, and the dynamic force required to hold the bow handle stationary. Estimates of the effect of air resistance were made and found to be less than 2% of the total system energy. The vibratory dynamics of the string and bow limbs subsequent to arrow exit was analyzed. The results of the elastic string considerations are in reasonable agreement with experimental data and negate the usual explanation for the long-standing discrepancy between theory and experiment as due to air resistance and hysteresis losses in the string and bow limbs.

Toxophilus. An thefe thynges althoughe they be trifles, yet bycaufe you be but a yonge fhoter, I would not leue them out.

Philologus. An fo you fhal do me mooft pleafure: The [bow] ftring I trow be the next.

Toxophilus. The next in dede. A thing though it be lytle, yet not a little to be regarded. But here in you mufte be contente to put youre trufte in honeft ftringers. And furely ftringers ought more diligently to be looked vpon by the officers than ether bower or fletcher, bycaufe they may deceyve a fimple man the more eafelyer.

Roger Ascham—*Toxophilus (1545)*

I. INTRODUCTION

Archery technology bloomed in the late 1920s paralleling the rapid advances of other sciences in that period. However, over the more than four centuries since Roger Ascham's[1] treatise on archery—an Aristotelian-like discussion between the teacher Toxophilus and student Philologus—no consideration was given to the effect that string elasticity might have on the performance of the bow and arrow system. The "honeft ftringer" has progressed from sinew, to flax, to dacron over the years, but no systematic analysis has been devoted toward understanding the consequences of higher string elasticity. The present paper attempts to eliminate that gap in our understanding.

The bow and arrow system provides a wealth of subjects amenable to scientific analyses. To illustrate this point, the following historical synopsis is presented.

Hickman began two decades of scientific studies of the sport of archery with the publication[2] in 1929 of his experimental studies of the velocity and acceleration of an arrow subsequent to its release by the archer. Hickman produced a number of articles concerning static strains and stresses in the bow, effects of a rigid midsection, static brace height effects, effect of bow length on arrow velocity, and a number of similar mechanics related studies. These articles are not available in the open literature, but were reproduced in a collection of works[3] on the technical aspects of archery published in a limited edition of 500 copies and available in some libraries.

Hickman began his studies on the classic English long bow, a favorite of archers of his time because of its much earlier success as a weapon in the Hundred Years War. Through his experimental and mathematical studies he soon came to the conclusion that the half-round cross sectional area of the long bow was not optimum for storing energy at minimum stress levels. This eventually resulted in the so-called flat bow, with limbs of rectangular cross-section and widths greater than thicknesses. It was also the predecessor of the modern laminated fiberglass and wood bow currently used in recreational archery. Hickman's studies culminated in 1937 with the publication[4] of his model of the long bow, which will be used in the present paper.

With an intensity similar to Hickman, Klopsteg began a series of studies circa 1930, culminating in a publication[5] in 1943 on the physics of bow and arrows. It was there that he proposed his concept of virtual mass, hinting that the mass effectively accelerated by the bow to the final arrow velocity is composed of the arrow mass plus one-third the mass of the string. The concept of virtual mass has been clarified in the present paper by identifying the explicit participation of both the string mass and limb mass in the dynamics of the bow and arrow system.

Klopsteg's studies crossed the gamut of the technical aspects of archery to which physical principles may be ap-

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plied: proper cross section for a bow limb, effects on scores of aiming errors, a study of instinctive and mechanical aiming techniques, the flight of the arrow, etc. Of particular interest were Klopsteg's studies of the so-called archer's paradox. The impulse normal to the axis of the arrow, caused by the release of the fingers from the string as well as the columnlike force of the string on the arrow during its acceleration, results in a significant bending of the arrow shaft as it transits the bow. This allows the arrow to undulate around the bow handle and follow a straight course towards a target without striking the bow handle. The ability of the arrow to follow such a straight course to the target requires that its bending mode period be matched to the time required by the arrow to exit the bow. When not properly matched, the arrow strikes the bow handle and is impulsively directed off its intended course. The asymmetrical (a symmetric arrangement would require the arrow to shoot through the center of the bow handle) relationship of the conventional bow and arrow, which results in an arrow flight toward a designated target, was a paradox to the archer. Klopsteg explained this paradox and provided a qualitative understanding of the reasons for matching arrows to a given bow and archer combination.

There are only five other published technical works of which the author is aware. Koeber[6] published an interesting study, unrelated to the physics of the bow and arrow in 1927. In that study Koeber considered the world distribution of types of arrow release techniques, i.e., one finger on the string, two fingers, three fingers, and various types of mechanical releases such as thumb rings and leather thongs. The purpose of this study was to shed some light on the migration of peoples about the planet Earth rather than understanding the dynamics of bows and arrows. English[7] published a study on the flight characteristics of arrows in 1930. Although he showed satisfactory agreement between his theoretical model and his extensive measurements of arrow trajectory parameters, the bases of his theoretical model can be easily attacked as unsound. He assumed, for example, that the drag force on the arrow was constant, depending only on the arrow's initial velocity. Higgins,[8] in a paper published in 1933, addressed the same subject using a more justifiable physical model, again with adequate agreement with the English experimental evidence, as well as his own. Schuster[9] published a model and numerical analysis of the modern working recurve bow in 1969. The working recurve bow is one in which the string partially winds itself around the bow limbs when the bow is in its relaxed or brace height state. Schuster assumed a massless, inextensible string in his study. Finally, in 1972, Stoylov *et al.*[10] published the results of some measurements of arrow velocity as a function of draw length. The basic purpose of this work was to describe two inexpensive methods for demonstrating the principles of conservation of momentum and energy to students in the poorly equipped schools of developing countries. One method involved the use of a ballistic pendulum made of banana stalk, the other the measurement of the free-fall range of an arrow released horizontally. The bow utilized in their study was of primitive design with an efficiency less than 50%.

Although some of the past works on the technical aspects of archery have considered the bow string from the points of view of its mass effect on efficiency, its brace height effects on arrow velocity, the number of strands required for a given bow weight (i.e., bow force at full draw), no comprehensive studies of the effect of an elastic string on bow and arrow performance appear to exist. None of the published works alludes to any potential importance of the elasticity of the string. It is the purpose of the present study to examine the effects of string elasticity. The importance of string elasticity can be appreciated from a summary of the results of the study described here: For an inextensible string the efficiency of a bow and arrow system has been shown to be given fairly accurately by the ratio of the arrow mass to the arrow mass plus one-third the mass of the string. For a typical arrow and string mass, this efficiency is 91%, i.e., the arrow carries off 91% of the energy initially stored in the bow by the effort made by the archer in pulling the bow to full draw. For the same arrow, bow limb, and bow string combination, but with an elastic string, the portion of the energy involved in the arrow flight is only 78% of the energy stored in the bow limbs and string at full draw; 11% is bound up in the kinetic energy of the limbs at arrow exit, 9% is in the kinetic energy of the string at arrow exit, and 2% is stored at arrow exit in potential energy in the string and bow limb system. The effect of air resistance is estimated to be less than 2% of the total energy at full draw. The air resistance losses were not included in the analysis that led to the partition of energy summarized above for the elastic string.

Before embarking on the course that will result in the numerics discussed above, consider the following qualitative description of the temporal behavior of a number of parameters important to bow and arrow dynamics. The parameters to be discussed are illustrated in Fig. 1. The solid line parameter profiles resulted from a numerical integration of the elastic string equations of motion. The dashed line parameter profiles are qualitative estimates, not drawn to scale. A parameter subscripted with the letter F signifies the value of that parameter at the full draw position of the arrow, the subscript B signifies a value at the brace height position (a braced bow is one which has been strung, but with no external forces applied; the brace height is the distance between the arrow nocking point on the string of a braced bow and the vertical midpoint of the outside face of the bow handle riser), and the subscript A denotes a value at arrow exit. Arrow exit is defined to occur when the arrow acceleration vanishes. The parameter x designates the position of the arrow nock or aft end of the arrow. As shown in Fig. 1, this parameter starts at the full draw position x_F, transits the brace height position x_B (because, during its dynamic motion, the string is stretched to a length greater than its brace height length) and, finally, arrow exit occurs at a position x_A, at the time t_A when the acceleration $\ddot{x}_A$ vanishes. Subsequent to arrow exit the arrow flies off with constant velocity as indicated by the solid curve in Fig. 1. The vibratory motion of the string after arrow exit is depicted by the dashed curves. Thus the dashed curve of the position x describes the motion of the center of the string subsequent to arrow exit. The center of the string continues to move in the direction of motion of the arrow until its velocity vanishes. Oscillations then begin in all variables except for the arrow, which has exited the bow. This vibratory motion is damped in a short time by air resistance, hysteresis losses, and losses of energy to the earth through the archer's body. The velocity profile, like the position profile, is a relatively smooth function of time leading to a final arrow velocity $\dot{x}_A$ and the subsequent vibration of the string as shown in Fig. 1. The acceleration, on the other hand,

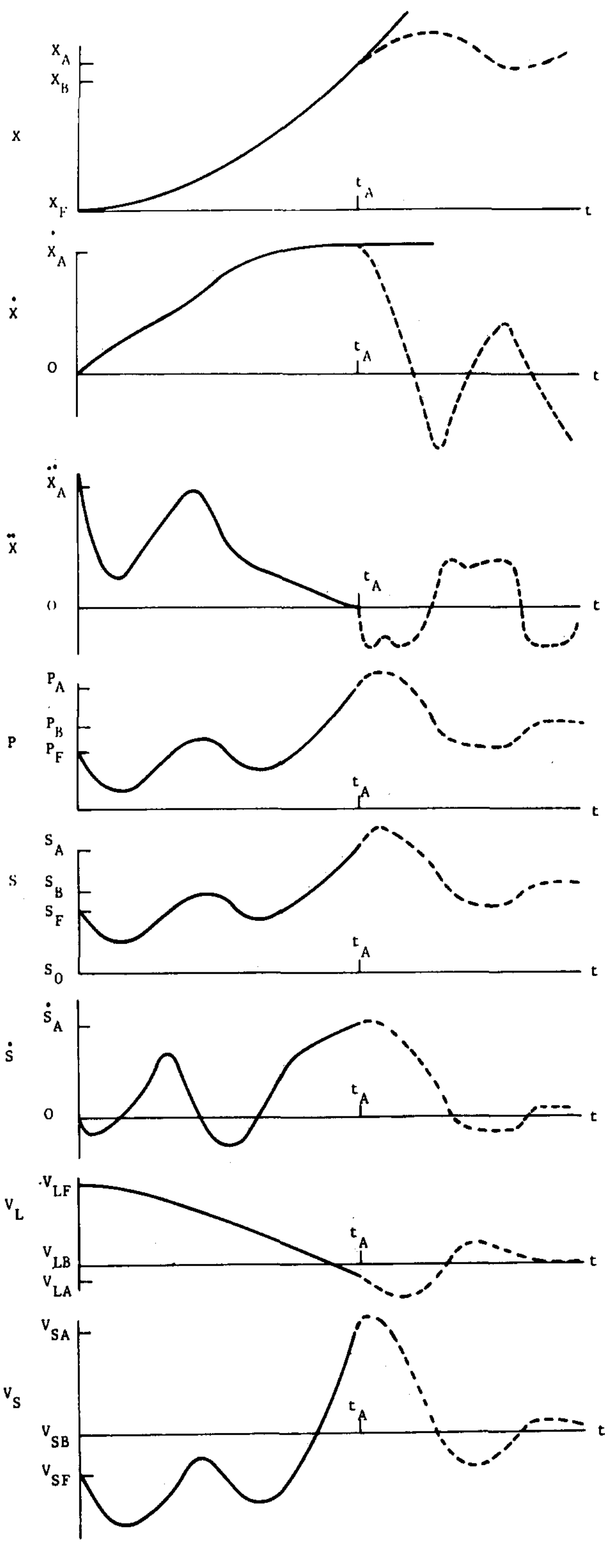

Fig. 1. Temporal profiles of a number of parameters important to bow and arrow dynamics. The subscripts A, B, and F refer to arrow exit, brace height position, and full draw, respectively; x is the displacement of the arrow, $\dot{x}$ the velocity, and $\ddot{x}$ the acceleration; P is the tension in the string; s and $\dot{s}$ the length of half the string and its time rate of change, respectively; V_L is the total potential energy associated with both bow limbs and V_S the total potential energy of both halves of the string due to its elastic elongation; t is the time after arrow release. The solid portion of the curves represent values of the parameters prior to arrow exit and were obtained from the elastic model results. The dashed curves represent bow and string parameters subsequent to arrow exit. The temporal profiles of the latter curves are qualitatively correct, but are not drawn to scale along either the ordinate or time axis.

undergoes some wild undulations following arrow release, as a competition for energy takes place among the arrow, string, and bow limbs. The acceleration finally decreases to zero at arrow exit. The string tension starts at a full draw value P_F which is, surprisingly, less than the static string tension P_B at brace height. The string tension then follows the acceleration profile fairly closely for some time, until it begins a dramatic increase as the string advances toward arrow exit. It is the experience of most archers that although a bow string might break, the arrow finds its way to the target apparently unaffected by the breakage. This course of events undoubtedly results from the fact that the peak string tension occurs after arrow exit. No unusual phenomena are observed in the variations of the string half-length s, velocity $\dot{s}$, or acceleration $\ddot{s}$, but the temporal profiles of the potential energies in the bow limbs V_L and string V_s deserve comment. Since arrow exit occurs at a time when the string is stretched to a length greater than the brace height length, the potential energy V_{s_A} of the string at arrow exit is larger than its potential energy V_{s_B} at the brace height. Conversely, at arrow exit the limbs are in a slightly more relaxed position with a potential energy V_{L_A} lower than the brace height potential V_{L_B}. If the increase in the potential energy of the string $V_{s_A} - V_{s_B}$ is greater than the decrease in the potential energy of the limbs $V_{L_A} - V_{L_B}$ (a negative value), then the total potential energy of the bow limb and string system, $V_{s_A} - V_{s_B} + V_{L_A} - V_{L_B}$, will be unavailable for arrow kinetic energy and the efficiency of the bow and arrow system will be reduced. This is indeed the case for the particular bow and arrow parameters addressed in the quantitative portion of this study.

The quantitative analysis of the events described above proceeds by first deriving the Lagrangian for a distributed mass bow, string, and arrow system of fairly general design. This is accomplished in Sec. II, where the equations of motion and the energy equation for the bow and arrow system are also derived. The Euler–Lagrange formulation is then utilized in Sec. III to generate an equivalent point-mass model and force diagrams for the Hickman model of a long bow. The point-mass model is used to derive expressions for the virtual mass of the bow, string tension, system momentum, and bow handle reaction force. The bow, arrow, and string parameters required for numerical solutions of the equations of motion are also described in Sec. III. In Sec. IV the solution of the equations of motion and energy for the case of the inextensible string are used to evaluate a number of parameters such as arrow position, arrow velocity, string tension, and efficiency at arrow exit. The subject of the vibratory motion of the bow subsequent to arrow exit is also treated in Sec. IV. The case of the elastic string is the subject of Sec. V. In order to obtain temporal profiles as well as the values of all interesting parameters at arrow exit, including the exit time, the equations of motion were numerically integrated. The results of the numerical integration are presented in Sec. V for the same bow and arrow system used for the inelastic string considerations. The subject of small vibrations of the bow and elastic string subsequent to arrow release is also discussed in Sec. V. The effects of air resistance and hysteresis losses are considered in Sec. VI and Appendix B. The conclusions of this analysis are presented in Sec. VII. The use of the bow and arrow as a teaching aid is discussed in Appendix A.

II. DISTRIBUTED MASS LAGRANGIAN FORMULATION

The Lagrangian for the bow and arrow system shown in Fig. 2 will be formed from the kinetic energy of the arrow (assuming that the mass of the arrow is localized at the center of the string), the kinetic energies of the string and limbs (arrived at assuming the string and limbs to have known mass distributions), and from the potential energies of the string and limbs. All motion will be assumed to take place in the plane of the bow and string, thus avoiding the complications of the archer's paradox. It will also be assumed that the bow and arrow system is symmetrical about the x axis. The latter axis coincides with the axis of the arrow. The Lagrangian will have two degrees of freedom which are taken to be the arrow position and the string halflength. In the coordinate system defined in Fig. 2, the kinetic energy of the arrow is

$$T_A = (1/2)M\dot{x}^2,$$

where M is the mass of the arrow.

The string will be assumed to have a uniform linear mass density λ which must vary with time, if the string is allowed to stretch. If s is the halflength of the string, then the mass of the string will be $m_s = 2\lambda s$. The point (x_s, y_s) on the string can be defined in terms of a variable ξ to be a fraction ϵ of the string halflength. Then $\xi = \epsilon s$, where ϵ is a constant independent of time. The string is assumed to remain straight under the dynamic forces it will experience. This assumption is tantamount to assuming infinite longitudinal and transverse signal velocities in the string. This assumption is discussed further in Sec. VII.

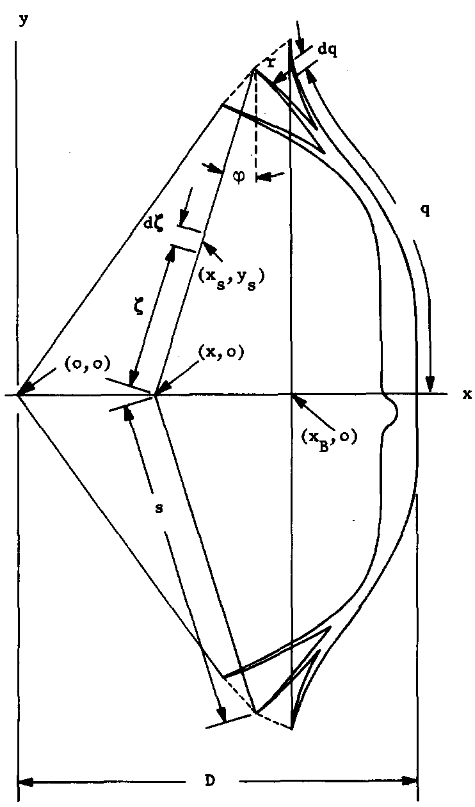

Fig. 2. Coordinate system for the Lagrangian formulation.

The kinetic energy of the string is

$$T_s = \int_0^1 \lambda s\, d\epsilon\, (\dot{x}_s^2 + \dot{y}_s^2)$$

where the string coordinates x_s and y_s are determined from Fig. 2 to be

$$x_s = x + \xi \sin\varphi \text{ and } y_s = \xi \cos\varphi,$$

so that $\dot{x}_s = \dot{x} + \epsilon s \dot{\varphi} \cos\varphi + \epsilon \dot{s} \sin\varphi$ and $\dot{y}_s = -\epsilon s \dot{\varphi} \sin\varphi + \epsilon \dot{s} \cos\varphi$. In terms of the two independent variables,

$$\dot{\varphi} = \varphi_x \dot{x} + \varphi_s \dot{s},$$

where a subscripted dependent variable denotes partial differentiation with respect to the independent variable. Substituting these relations into the expression for the string kinetic energy and carrying out the integration, the string kinetic energy is found to be a quadratic in the velocities of the independent variables,

$$T_s = \frac{1}{2} m_s \left[\left(1 + s\varphi_x \cos\varphi + \frac{1}{3} s^2 \varphi_x^2\right)\dot{x}^2 + \left(\sin\varphi + s\varphi_s \cos\varphi + \frac{2}{3} s^2 \varphi_x \varphi_s\right)\dot{x}\dot{s} + \frac{1}{3}\left(1 + s^2 \varphi_s^2\right)\dot{s}^2\right].$$

It will be possible to provide a general formulation of the kinetic energy of the limb, but, to go beyond the expressions derived below, the configuration of the limbs must be specified as done, for example, in Sec. III for the Hickman model. The mass of either of the two limbs can be obtained from

$$m_L = \int_{\text{limb}} dq\, \rho(q)\tau(q)w(q),$$

where q is defined in Fig. 2, ρ is the volume mass density of the limb, τ the thickness, and w the width. Obviously, an even more complicated expression would be necessary, if one were to consider the composition of the limb in more detail, e.g., a laminated limb of wood and fiberglass. The approach employed here assumes an average over such complexities. The kinetic energy of the limbs then becomes

$$T_L = \frac{1}{2} \int_{\text{limbs}} dq\, \rho(q)\tau(q)w(q)[\dot{r}(x,s,q)]^2,$$

where r is displacement of the limb element dq (see Fig. 2). Now $\dot{r} = r_x \dot{x} + r_s \dot{s}$, so that the kinetic energy of the limbs assumes a quadratc form in the velocities of the independent variables,

$$T_L = \frac{1}{2} \int_{\text{limbs}} dq\, \rho \tau w (r_x^2 \dot{x}^2 + 2 r_x r_s \dot{x}\dot{s} + r_s^2 \dot{s}^2).$$

The potential energy of the string, again assuming infinite signal velocities, can be put into the form

$$V_s = 2k(s - s_0)^2 - 2k(s_B - s_0)^2,$$

where k is the elastic constant for the entire string of length $2s$, s_0 is the unstretched string halflength, and s_B is the string halflength at brace height. The potential energy of the string at brace height has been chosen to be zero. Note that the potential energy of the string is independent of the arrow position, i.e., the variable x.

The potential energy of the limbs, just as its kinetic energy, cannot be detailed without specifying the characteristics of the limbs. However, it will have the general form

$$V_L = V_L(x,s) - V_L(x_B,s_B).$$

Again, the zero of potential energy is taken to be at the brace height position.

The Lagrangian for the bow and arrow system can now be written as

$$L = \frac{1}{2}(M + K_1)\dot{x}^2 + \frac{1}{2}K_2\dot{x}\dot{s} + \frac{1}{2}K_3\dot{s}^2 - V_L - V_S,$$

where the virtual masses of the system are

$$K_1 = m_s\left(1 + s\varphi_x\cos\varphi + \frac{1}{3}s^2\varphi_x^2\right) + 2\int_{\text{limb}} dq\,\rho\tau w r_x^2,$$

$$K_2 = m_s\left(\sin\varphi + s\varphi_s\cos\varphi + \frac{2}{3}s^2\varphi_x\varphi_s\right) + 4\int_{\text{limb}} dq\,\rho\tau w r_x r_s,$$

and

$$K_3 = \frac{1}{3}m_s(1 + s^2\varphi_s^2) + 2\int_{\text{limb}} dq\,\rho\tau w r_s^2.$$

Note that the integral over limb is half the same integral over limbs.

The Euler–Lagrange equations of motion are readily determined from the Lagrangian to be

$$(M + K_1)\ddot{x} + \frac{1}{2}K_2\ddot{s} + \frac{1}{2}K_{1x}\dot{x}^2 + K_{1s}\dot{x}\dot{s} + \frac{1}{2}(K_{2s} - K_{3x})\dot{s}^2 + V_{Lx} = 0,$$

and

$$\frac{1}{2}K_2\ddot{x} + K_3\ddot{s} + \frac{1}{2}(K_{2x} - K_{1s})\dot{x}^2 + K_{3x}\dot{x}\dot{s} + \frac{1}{2}K_{3s}\dot{s}^2 + V_{Ls} + V_{ss} = 0.$$

The first integral of the equations of motion produces the energy equation

$$\frac{1}{2}(M + K_1)\,\dot{x}^2 + \frac{1}{2}K_2\dot{x}\dot{s} + \frac{1}{2}K_3\dot{s}^2 + V_L + V_s = E_F,$$

where E_F is total energy stored in the bow limbs and string at full draw.

In order to proceed one must choose a specific model of the bow limbs. This will be done in Sec. III.

III. POINT MASS MODEL FOR THE LONG BOW

Hickman[3] modelled the long bow by replacing the working portions of the bow limbs with a straight, inflexible limb that rotated about a point at one end of the limb under a restoring force. This model is depicted in Fig. 3. Each working limb is assumed to be symmetrical about the x axis and of length l. The stationary portion of the bow, the handle riser, is also symmetrical about the x axis and of length $2L$. As indicated in Fig. 3, the position of the limb is characterized by the angle θ. Consider the limbs to have a constant mass density ρ, constant thickness τ, and to vary in width uniformly from a width $w_0 + w_1$ at the hinge point to a width w_0 at the tip or limb nocking point. The mass of a single working limb is then,

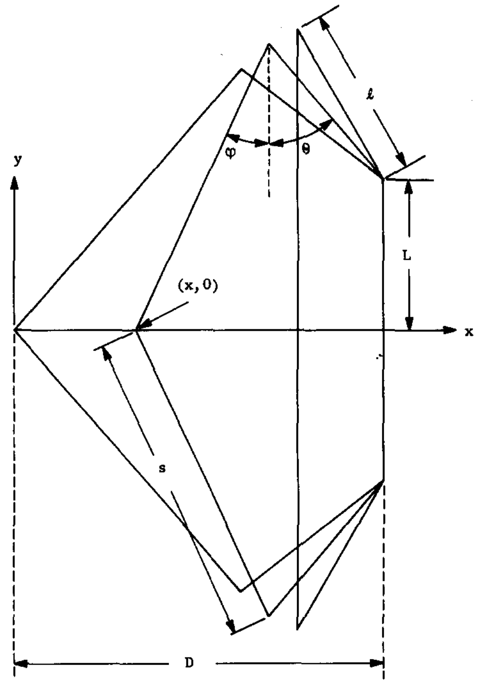

Fig. 3. Hickman stick model of the long bow.

$$m_L = \rho\tau w_0\int_0^l dq\left(1 + \frac{w_1}{w_0} - \frac{w_1}{w_0}\frac{q}{l}\right) = \rho\tau w_0 l\left(1 + \frac{1}{2}\frac{w_1}{w_0}\right).$$

The parameter $r = q\theta$, so that the kinetic energy of the limbs becomes

$$T_L = \rho\tau w_0\int_0^l dq\left(1 + \frac{w_1}{w_0} - \frac{w_1}{w_0}\frac{q}{l}\right)q^2\dot{\theta}^2.$$

Then, performing the integral with

$$\dot{\theta} = \theta_x\dot{x} + \theta_s\dot{s},$$

the kinetic energy of the limbs can be written

$$T_L = m_L\beta l^2(\theta_x^2\dot{x}^2 + 2\theta_x\theta_x\dot{x}\dot{s} + \theta_s^2\dot{s}^2),$$

where

$$\beta = \frac{1}{3}\left(1 + \frac{1}{4}\frac{w_1}{w_0}\right)\bigg/\left(1 + \frac{1}{2}\frac{w_1}{w_0}\right).$$

Combining this expression with the relationship found earlier for the kinetic energy of the string, one finds the following results for the virtual masses of the bow system that are to be used in the equation of motion,

$$K_1 = m_s\left(1 + s\varphi_x\cos\varphi + \frac{1}{3}s^2\varphi_x^2\right) + 2\,m_L\beta l^2\theta_x^2,$$

$$K_2 = m_s\left(\sin\varphi + s\varphi_s\cos\varphi + \frac{2}{3}s^2\varphi_x\varphi_s\right) + 4\,m_L\beta l^2\theta_x\theta_s,$$

$$K_3 = \frac{1}{3}m_s(1 + s^2\varphi_s^2) + 2m_L\beta l^2\theta_s^2.$$

The potential energy of the limbs in the Hickman model[4] is of the form

$$V_L = \frac{1}{2}\kappa s_0 l \frac{\sin\theta_0}{\theta_0}(\theta^2 - \theta_B^2),$$

where θ_0 is the value of θ for a string of unstretched halflength s_0, and θ_B is the value of θ at the brace height for an elastic string. The parameter κ is the spring constant of the limbs and, as mentioned earlier, l is the length of the working portion of the bow limb. This form for the potential energy of the limbs is discussed in greater detail later in this section.

Substituting the above expressions for the virtual masses and potential energies into the equations of motion, it becomes apparent after a fairly extensive rearrangement of the terms, that the equations of motion can be interpreted as illustrated in the point-mass model shown in Fig. 4. This point-mass model is obtained by placing the arrow mass and one-fourth the string mass at the center of the string. The force acting on that total mass is then the reaction force $[M + (1/4)m_s]\ddot{x}$ and the string tension force $(1/2)dV_s/ds$ directed along the string for each string half as shown in Fig. 4. Three-eighths of the string mass is placed two-thirds of the string halflength on each side of the string center. The forces acting on those masses involve a centrifugal force $(3/8)m_s(2/3)s\dot{\varphi}^2$, a reaction force $(3/8)m_s(2/3)s\ddot{\varphi}$, a reaction force $(3/8)m_s(2/3)\ddot{s}$, and a Coriolis force $2(3/8)m_s(2/3)\dot{s}\dot{\varphi}$. A mass $m_L' = m_L(1 + 6\beta)^2/36\beta$ is to be placed on each limb a distance $l' = 6\beta l/(1 + 6\beta)$ from the respective hinge points. Torques, as illustrated in Fig. 4, are to act about the hinge; $m_L' l'^2\ddot{\theta}$ due to the reaction force on the masses m_L', and $(1/2)dV_L/d\theta$ due to the restoring force in each of the limbs. Finally, a centrifugal force $m_L' l'\dot{\theta}^2$ acts on the masses m_L', while the string tension $(1/2)dV_s/ds$ acts along the string on each bow tip. Note that the moment of inertia of each limb is $m_L' l'^2 = m_L\beta l^2$.

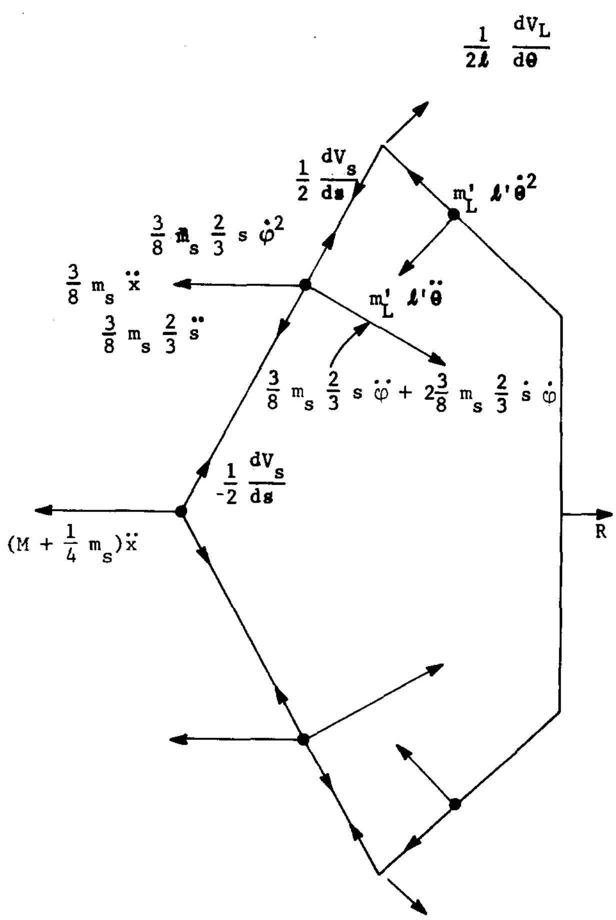

Fig. 4. Point mass model of the Hickman long bow.

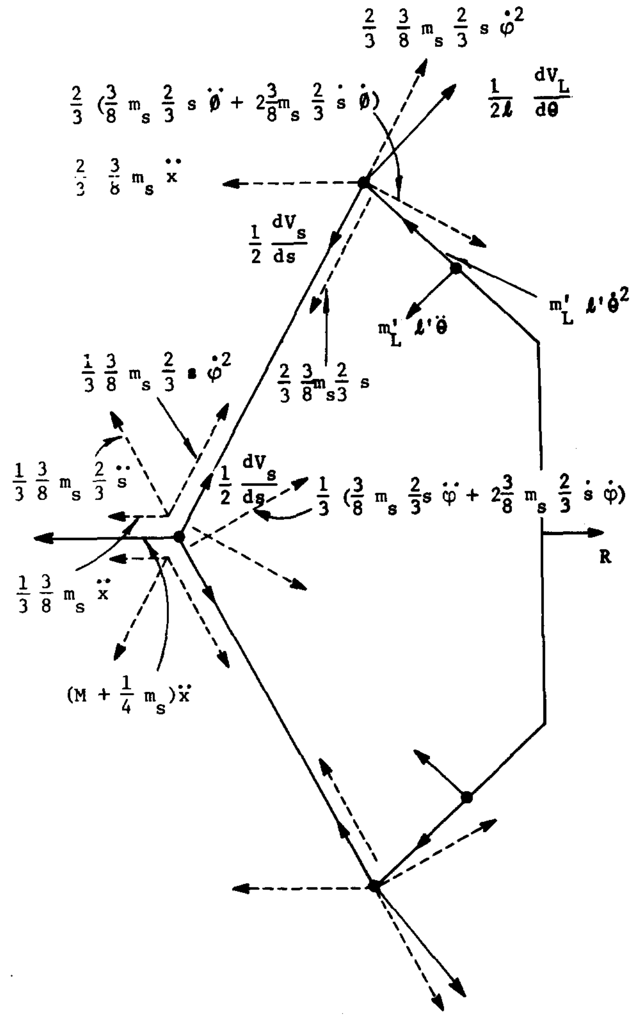

Fig. 5. Rearrangement of forces preserving an overall force and torque balance.

In Fig. 5 the forces have been rearranged while preserving an overall force and torque balance to allow a ready determination of the system momentum p_x in the x direction, bow handle reaction force R (i.e., the force required to maintain the bow handle stationary), and the string tension P. These quantities are found to be, after some algebraic simplifications,

$$p_x = (M + m_s)\dot{x} + (1/2)m_s(s\dot{\varphi}\cos\varphi + \dot{s}\sin\varphi) - 2m_L' l'\dot{\theta}\cos\theta,$$

$$R = (M + m_s)\ddot{x} + (1/2)m_s(s\ddot{\varphi}\cos\varphi + \ddot{s}\sin\varphi + 2\dot{s}\dot{\varphi}\cos\varphi - s\dot{\varphi}^2\sin\varphi) - 2m_L' l'(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta),$$

$$P = \frac{1}{2}\frac{dV_s}{ds} = 2k(s - s_0),$$

or

$$P = \left[\left(M + \frac{1}{2}m_s\right)\ddot{x} + \frac{1}{6}m_s\left(\ddot{s}\sin\varphi - \dot{\varphi}^2\sin\varphi\right.\right.$$

$$+ s\ddot{\varphi}\cos\varphi + 2\dot{s}\dot{\varphi}\cos\varphi\Big)\Big]\Big/ 2\sin\varphi,$$

or

$$P = \left[\frac{1}{2l}\frac{dV_L}{d\theta} - \frac{1}{6}m_s\left(\frac{3}{2}\ddot{x}\cos\theta + \ddot{s}\sin(\theta+\varphi) + (s\ddot{\varphi} + 2\dot{s}\dot{\varphi})\cos(\theta+\varphi) + s\dot{\varphi}^2\sin(\theta+\varphi)\right) + m_L\beta l\ddot{\theta}\right]\Big/\sin(\theta+\varphi).$$

The first expression for the string tension follows directly from its potential energy, the second from a balance of forces at the center of the string, and the third from a balance of torques about the upper limb hinge point.

The partial derivatives of the angles θ and φ with respect to the independent variables x and s are required to determine the virtual masses, derivatives of the virtual masses, and the derivatives of the limb potential energy. The requisite relationships can be obtained from the geometry of the Hickman model shown in Fig. 3. Thus

$$D - x = l\sin\theta + s\sin\varphi,$$

$$L + l\cos\theta = s\cos\varphi.$$

The first-order derivatives are

$$\varphi_x = -\frac{\sin\theta}{s\sin(\theta+\varphi)}, \quad \varphi_s = \frac{\cos(\theta+\varphi)}{s\sin(\theta+\varphi)},$$

$$\theta_x = -\frac{\sin\varphi}{l\sin(\theta+\varphi)}, \quad \theta_s = -\frac{1}{l\sin(\theta+\varphi)},$$

and the higher-order derivatives can be obtained by partial differentiation of these expressions.

The final task of this section will be to delineate the physical characteristics of the bow and arrow system that will be used for numerical calculations in Secs. IV and V. In order to obtain numerical solutions, 10 parameters must be defined for the Hickman model.

We begin by specifying the arrow mass to be $M = 25$ g (385 grain). This is a mass typical of a 70-cm (28-in.) length, aluminum tubular arrow of outside diameter 0.794 cm (20/64 in.) and 0.0330-cm (0.013-in.) wall thickness, popularly designated as a 20-13 arrow.

The mass of a typical string will be taken to be $m_s = 7$ g (108 grain). Strings are normally whipped near the center and at the ends, but the string will be assumed to have uniform linear density. The string will be assumed to have an unstretched halflength $s_0 = 85$ cm (full length 67 in.). Sneler[11] has measured the elastic constants for a number of strings, resulting in a typical value to be used here of 7000 N/m (50 lb/in.). This is characteristic of a 10-strand dacron string of the length assumed here.

The mass of a typical bow limb will be taken to be $m_L = 93$ g (1440 grain). This value was arrived at by using the measured mass density $\rho = 1.4$ g/cm^3 of a short section of a limb broken from a fiberglass–wood laminated bow, and with typical values of thickness $\tau = 0.865$ cm, width at the tip $w_0 = 1.0$ cm, aspect ratio $w_1/w_0 = 1.2$, and working limb length $l = 48$ cm. The aspect ratio of 1.2 yields a value $\beta = 0.271$. We will use a value of $\theta_0 = 30°$ as the angle made by the working limb with the vertical for a string of unstretched length. The full draw length will be the length of the arrow of $D = 70$ cm (28 in.). Finally, the sping constant of the bow must be chosen. The Hickman model assumed the potential energy of the limbs to be proportional to the square of the deflection and this results in reasonable fits to the force-draw curves of the bow. The potential energy of the limbs is assumed to vanish at the brace height. Consequently, the potential must have the form

$$V_L = C(\theta^2 - \theta_B^2).$$

The spring constant will be evaluated in terms of the value it would have for an inelastic string of halflength s_0. The spring constant is

$$\kappa = \frac{d^2V_L}{dx^2}\bigg|_{x=x_0},$$

where the derivative is evaluated at $s = s_0$, $\varphi = 0$, and $\theta = \theta_0$. Now

$$\frac{d^2V_L}{dx^2} = 2C(\theta\theta_{xx} + \theta_x^2),$$

so, evaluating the derivatives from the geometrical relationship presented earlier, the constant is found to be

$$C = (1/2)\kappa s_0 l \sin\theta_0/\theta_0,$$

and the potential takes the form

$$V_L = \frac{1}{2}\kappa s_0 l \frac{\sin\theta_0}{\theta_0}(\theta^2 - \theta_B^2).$$

Fitting this expression to measured values for typical bows results in the spring constant of 620 N/m (3.5 lb/in.) to be used for the present calculations.

The ten parameters defined above, which must be specified in order to carry out numerical integrations of the equations of motion, are summarized in Table I.

IV. INEXTENSIBLE STRING CONSIDERATIONS

Before tackling the more complex problem of bow and arrow dynamics with an elastic string, let us first consider the inextensible string problem. This will be useful as a base against which the results of the elastic model can be compared, as well as providing a point of departure for reviewing past works.

If the string is inelastic, the energy equation and equations of motion reduce to the following simple expressions,

$$(1/2)(M + K)\dot{x}^2 + V_L = E_F$$

and

$$(M + K)\ddot{x} + (1/2)K_x\dot{x}^2 + V_{Lx} = 0,$$

Table I. Typical physical constants for the bow and arrow system.

M	Arrow mass	25 g (385 grain)
m_s	String mass	7 g (108 grain)
s_0	String halflength	85 cm (33.5 in.)
k	String elastic constant	7000 N/m (40 lb/in.)
m_L	Limb mass	93 g (1440 grain)
l	Limb length	48 cm (19 in.)
β	Moment of inertia factor	0.271
D	Full draw length	70 cm (28 in.)
κ	Bow spring constant	620 N/m (3.5 lb/in.)
θ_0	Bow limb angle	30°

where the virtual mass is

$$K = m_s[1 + s_0\varphi_x \cos\varphi + (1/3)s_0^2\varphi_x^2] + 2m_L\beta l^2\theta_x^2.$$

Let the subscript A denote values of the variables at arrow exit, which occurs when the acceleration of the arrow vanishes, i.e., when $\ddot{x}_A = 0$, and the arrow separates from the string. Let $\varphi_A = \Delta\varphi$ and $\theta_A = \theta_0 + \Delta\theta$. The geometry equations then yield to the lowest order in $\Delta\varphi$:

$$\Delta\theta = s_0(\Delta\varphi)^2/2l\sin\theta_0.$$

The derivatives of θ and φ required to evaluate the virtual mass derivative K_{xA} at arrow exit are found to be

$$\varphi_{xA} = -(1 - \Delta\varphi\cos\theta_0/\sin\theta_0)/s_0,$$

$$\theta_{xA} = -\Delta\varphi/l\sin\theta_0,$$

$$s_0^2\varphi_{xxA} = -\frac{\cos\theta_0}{\sin\theta_0} + \Delta\varphi(1 + 3\cos^2\theta_0/\sin^2\theta_0)$$

$$s_0 l\theta_{xxA} = (1 - 3\Delta\varphi\cos\theta_0/\sin\theta_0)/\sin\theta_0,$$

while the potential energy derivative becomes

$$V_{LxA} = -\kappa s_0\Delta\varphi.$$

Putting these expressions into the equation of motion, the angle made by the string with the vertical at arrow exit is found to be

$$\Delta\varphi = -\left(\frac{m_s\dot{x}_A^2\cos\theta_0}{6s_0\sin\theta_0}\right)\Bigg/\left(\kappa s_0 + \frac{2m_L\beta\dot{x}_A^2}{s_0\sin^2\theta_0} - \frac{m_s\dot{x}_A^2}{6s_0\sin^2\theta_0}(1 - 3\sin^2\theta_0)\right).$$

The efficiency of the bow and arrow system is defined to be the ratio of the kinetic energy of the arrow at arrow exit to the energy stored in the bow by the archer in pulling the bow to full draw. Thus

$$\xi = (1/2)M\dot{x}_A^2/E_F.$$

Using the energy equation, the efficiency can be written to lowest-order terms in $\Delta\varphi$ as

$$\xi = (M/\mathcal{M})(1 - \kappa s_0^2\Delta\varphi^2/2E_F),$$

where

$$\mathcal{M} = M + m_s(1 + \Delta\varphi\cot\theta_0)/3 + 2\,m_L\beta\Delta\varphi^2\csc^2\theta_0.$$

The energy at full draw is given by

$$E_F = \frac{1}{2}\kappa s_0 l\frac{\sin\theta_0}{\theta_0}(\theta_F^2 - \theta_0^2),$$

where θ_F and φ_F are determined from geometry:

$$D = l\sin\theta_F + s_0\sin\varphi_F,$$

$$l(\cos\theta_0 - \cos\theta_F) = s_0(1 - \cos\varphi_F).$$

Using the bow and arrow parameters listed in Table I, one finds from the above equations, the following numerical results:

$$\xi = 0.9150, \quad \Delta\varphi = -5.89\times10^{-3}, \quad \dot{x}_A = 56.0\ \text{m/sec (184 ft/sec)}.$$

Other quantities of interest are the string tension

$$P_A = \frac{1}{2}\kappa s_0 + \frac{m_s\dot{x}_A^2}{6s_0\sin^2\theta_0} + \frac{m_L\beta\dot{x}_A^2}{s_0\sin^2\theta_0} = 653\ \text{N (147 lb)};$$

the component of momentum in the x direction,

$$p_{xA} = \left(M + \frac{1}{2}m_s\right)\dot{x}_A - \frac{m_s\dot{x}_A\cos\theta_0}{2\sin\theta_0}\Delta\varphi - \frac{2m_L'l'\dot{x}_A\cos\theta_0}{l\sin\theta_0}\Delta\varphi = 1.645\ \text{kg m/sec};$$

the reaction force on the bow handle,

$$-R_A = \frac{m_s\dot{x}_A^2\cot\theta_0}{2s_0}\left(1 + \frac{4m_L'l'}{m_s l}\right) = 542\ \text{N (122 lb)};$$

and the following:

$$\theta_F = 0.793, \quad \varphi_F = 0.435, \quad E_F = 42.9\ \text{J}.$$

It is interesting to note that slightly before arrow exit, when the arrow is at the brace height and $\varphi = 0$, then the limbs are stationary ($\theta_x = 0$) and the fraction of the total energy E_F in the kinetic energy of the arrow can readily be determined from the energy equation to be $M/[M + (1/3)m_s] = 0.9146$, only slightly smaller than the final arrow exit efficiency of 0.9150.

Subsequent to arrow exit from the bow, an amount of energy $(1 - \xi)E_F$ remains in the bow limbs and string, causing a vibratory motion which is eventually damped by air resistance, hysteresis losses, and transfer of energy through the archer's body to the earth. For the case of the inelastic string the energy equation and the equation of motion are then,

$$(1/2)K\dot{x}^2 + V_L = (1 - \xi)E_F,$$

$$K\ddot{x} + (1/2)K_x\dot{x}^2 + V_{Lx} = 0.$$

For very small displacements, the equation of motion reduces to (neglecting dissipation)

$$(1/3)m_s\ddot{x} + \kappa(x - x_0) = 0,$$

and the vibration frequency is

$$\omega_0 = \sqrt{3\kappa/m_s} \quad \text{or} \quad f_0 = 82\ \text{Hz}.$$

If the first-order nonlinear terms are retained, the first integral of the equation of motion can be written in the dimensionless form

$$\left(1 - z\frac{\cos\theta_0}{\sin\theta_0} + z^2\frac{6m_L\beta}{m_s\sin^2\theta_0}\right)\left(\frac{dz}{d\tau}\right)^2 = \frac{2(1-\xi)E_F}{\kappa s_0^2} - z^2,$$

where $z = (x - x_0)/s_0$ and $\tau = \omega_0 t$. Numerical integration of this equation gives a frequency

$$\omega = 0.80\,\omega_0 \quad \text{or} \quad f = 66\ \text{Hz}.$$

The amplitude of the vibration is

$$z^2 = 2(1 - \xi)E_F/\kappa s_0^2 = (0.128)^2$$

or $x - x_0 = 10.8$ cm.

V. ELASTIC STRING CONSIDERATIONS

The theoretical approach used in Sec. IV to analyze the dynamics of the bow and arrow for the case of an inelastic string, specifically the technique of evaluating the equations of motion and energy at arrow exit where the acceleration $\ddot{x}_A$ vanishes, allows one to determine all the dynamic parameters of interest at the time of arrow exit with the single exception of the exit time t_A itself. For the case of the elastic string the equations of motion were numerically integrated to arrive at values of the efficiency, arrow velocity, string tension, and other parameters of interest at arrow exit. The numerical integration provides, in addition, the time required for the arrow to exit the bow and temporal profiles of all variables. The results of such a numerical integration are summarized in this section.

The arrow, bow limb, and string parameters listed in Table I were used in the calculation. For the elastic string calculations, the handle riser halflength L was determined, assuming an inelastic string, from

$$L = s_0 - l\cos\theta_0 = 43.43 \text{ cm}.$$

Since the string is elastic, the relations determining the static conditions are not the same as for the inextensible string. For example, the brace height parameters must be iterated from the equations

$$s_B = s_0\left(1 + \frac{\kappa}{4k}\frac{\sin\theta_0}{\theta_0}\frac{\theta B}{\sin\theta_B}\right) = 86.86 \text{ cm},$$

$$\theta_B = \cos^{-1}[(s_B - L)/l] = 0.4401.$$

The static tension in the string at the brace height is

$$P_B = 2k(s_B - s_0) = 260 \text{ N } (58 \text{ lb}).$$

At full draw an iteration of three equations is required to determine the static half-string length and the vertical angles θ_F and φ_F,

$$s_F = s_0\left(1 + \frac{\kappa}{4k}\frac{\sin\theta_0}{\theta_0}\frac{\theta_F}{\sin(\theta_F + \varphi_F)}\right) = 86.47 \text{ cm},$$

$$\theta_F = \tan^{-1}\left(\frac{L}{D}\right) + \sin^{-1}\left(\frac{D^2 + L^2 + l^2 - s_F^2}{2l\sqrt{D^2 + L^2}}\right) = 0.7608,$$

$$\varphi = \tan^{-1}\left(\frac{D - l\sin\theta_F}{L + l\cos\theta_F}\right) = 0.4409.$$

The static string tension, force that the archer must exert at the string center (popularly known as the bow weight), and the energy stored in the limbs and string are determined from

$$P_F = 2k(s_F - s_0) = 205 \text{ N } (46 \text{ lb}),$$

$$F_F = 2P_F\sin\varphi_F = 175.2 \text{ N } (39 \text{ lb}),$$

$$E_F = \frac{1}{2}\kappa s_0 l\frac{\sin\theta_0}{\theta_0}(\theta_F^2 - \theta_B^2) + 2k(s_B - s_0)^2 = 44.7 \text{ J } (33 \text{ ft lb}).$$

For the purposes of numerical integration the two equations of motion are solved for $\ddot{x}$ and $\ddot{s}$ in the form

$$\ddot{x} = [(1/2)K_2F_2 - K_3F_1]/[(M + K_1)K_3 - (1/4)K_2^2]$$

and

$$\ddot{s} = [(1/2)K_2F_1 - (M + K_1)F_2]/[(M + K_1)K_3 - (1/4)K_2^2],$$

where

$$F_1 = (1/2)K_{1x}\dot{x}^2 + K_{1s}\dot{x}\dot{s} + (1/2)(K_{2s} - K_{3x})\dot{s}^2 + V_{Lx}$$

and

$$F_2 = (1/2)(K_{2x} - K_{1s})\dot{x}^2 + K_{3x}\dot{x}\dot{s} + (1/2)K_{3s}\dot{s}^2 + V_{Ls} + V_{ss}.$$

A simple Newtonian integration proved to have adequate accuracy for the calculation. The initial conditions were taken to be

$$x(0) = 0, \quad \dot{x}(0) = 0,$$

$$s(0) = s_F, \quad \dot{s}(0) = 0.$$

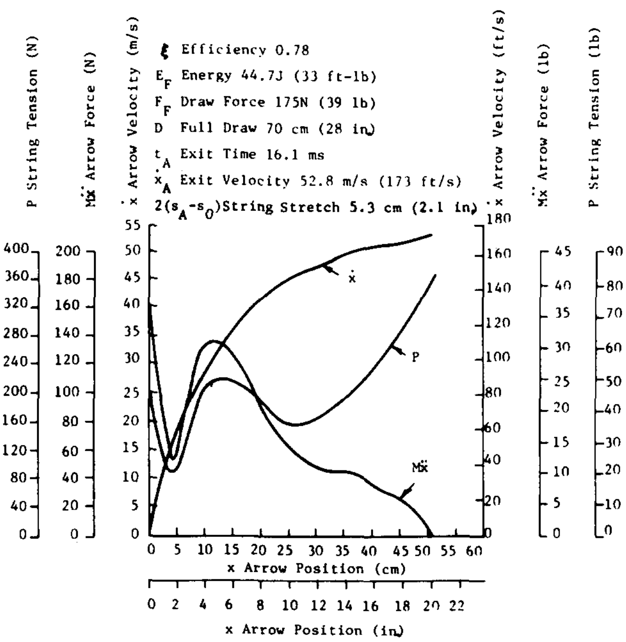

Fig. 6. Temporal profiles of the arrow velocity, force, and the string tension.

Temporal profiles of some of the more interesting parameters are shown in Figs. 6 and 7. Instantaneous values of a number of parameters are listed in Table II. A comparison of the inelastic calculations of Sec. IV and the elastic calculations of the present section can be made by utilizing Table III, where the parameters for both cases are tabulated.

Table II. Numerical integration results for the elastic string.

Physical characteristics of the arrow, string, and bow
M = 25 g (385 grain), D = 70 cm (28 in.)
m_s = 7 g (108 grain), s_0 = 85 cm (33.5 in.)
κ = 620 N/m (3.5 lb/in.)
m_L = 93 g (1440 grain), l = 48 cm (18.9 in.)
k = 7000 N/m (40 lb/in.)
β = 0.271, θ_0 = 0.5236 (30°), L = 43.4 cm (17.1 in)
Brace height parameters
P_B = 260 N (58 lb), s_B = 86.9 cm (34.2 in.)
$2(s_B - s_0)$ = 3.72 cm (1.46 in.)
θ_B = 0.4401 (25.2°), $D - x_B$ = 20.4 cm (8.0 in.)
Full draw parameters
P_F = 205 N (46 lb), s_F = 86.5 cm (34 in.)
$2(s_F - s_0)$ = 2.94 cm (1.16 in.)
F_F = 175 N (39 lb), θ_F = 0.7608 (43.6°), φ_F = 0.4409 (25.3°)
E_F = 44.7 J (33 ft lb)
Arrow exit parameters
t_A = 16.1 msec, x_A = 51.5 cm (20.3 in.), $x_A - x_B$ = 1.94 cm (0.76 in.)
$\dot{x}_A$ = 52.8 m/sec (173 ft/sec), s_A = 87.6 cm (34.5 in.) $2(s_A - s_0)$ = 5.3 cm (2.1 in.)
$\dot{s}_A$ = 5.5 m/sec (18 ft/sec), θ_A = 0.4006 (23.0°)
$\varphi_A = -1.92 \times 10^{-3}$ (-3.4×10^{-5} deg)
ξ_A = 0.781, V_L = −4.0 J (−3.0 ft lb), V_s = 4.9 J (3.6 ft lb)
P_A = 369 N (83 lb), R_A = −54 N (−12 lb), P_{x_A} = 2.1 kg m/s
$\dot{\theta}_A$ = −29.1/sec, $\ddot{\theta}_A$ = 3025/sec^2, $\dot{\varphi}_A$ = −45.6/sec, $\ddot{\varphi}_A$ = −771/sec^2

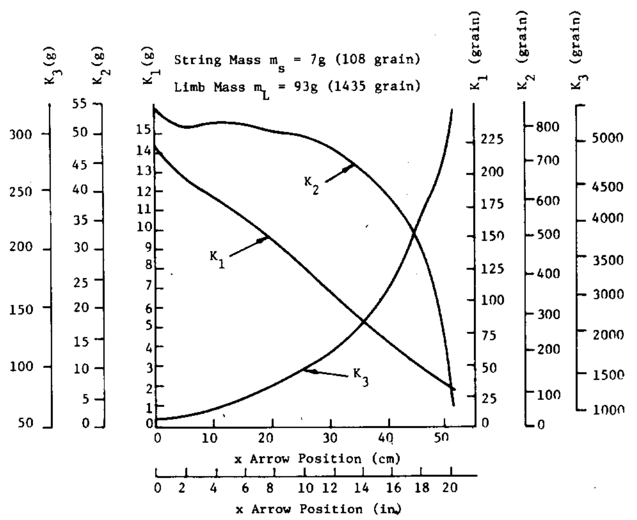

Fig. 7. Temporal profiles of the virtual masses.

The partition of energy among the arrow, bow limbs, and string can be determined from the results of the present section. The kinetic energies of the arrow, string, and limbs are, respectively,

$$T_M = (1/2)M\dot{x}^2,$$

$$T_s = (1/2)m_s\{\dot{x}^2 + s\dot{x}\dot{\varphi}\cos\varphi + \dot{x}\dot{s}\sin\varphi + (1/3)[s^2\dot{\varphi}^2 + \dot{s}^2]\},$$

$$T_L = m_L\beta l^2\dot{\theta}^2.$$

Table III. Comparison of inelastic and elastic parameters.

Parameter	Inelastic string	Elastic string
ξ_A	0.9150	0.7810
φ_A	-5.89×10^{-3}	-1.925×10^{-3}
θ_A	0.5237	0.4006
$\dot{x}_A$	56.0 m/sec	52.8 m/sec
P_A	653 N	369 N
px_A	1.645 kg m/sec	2.10 kg m/sec
R_A	−542 N	−54 N
E_F	42.9 J	44.7 J
F_F^{static}	178 N	175 N
θ_F	0.7934	0.7608
φ_F	0.4346	0.4409
$M\ddot{x}_F/F_F^{\text{stat}}$	0.648[a]	0.934[a]
P_F/P_F^{stat}	0.717[a]	1.000[a]

[a] The footnoted results are obtained from the dynamic expressions for the force on the arrow $M\ddot{x}_F$ and the string tension P_F at arrow release ratioed to the static expressions for the full draw force and string tension, respectively. These ratios are

$$M\ddot{x}_F/F_F^{\text{stat}} = M/(M + K_F)$$

and

$$P_F/P_F^{\text{stat}} = \left(M + \frac{1}{2}m_s + \frac{1}{6}m_s s_0\,\varphi_{xF}\cos\varphi_F\right)\Big/(M + K_F)$$

for the inelastic string, and

$$M\ddot{x}_F/F_F^{\text{stat}} = -M\left(\frac{1}{2}K_{2F}(V_{LsF} + V_{ssF}) - K_{3F}V_{LxF}\right)\Big/\left[(M + K_{1F})K_{3F} - \frac{1}{4}K_{2F}^2\right]V_{LxF}$$

and $P_F/P_F^{\text{stat}} = 1$ for the elastic string.

Evaluating these expressions at arrow exit, we find the following results for the kinetic and potential energies. All quantities used in the evaluation are presented in Table II. The energies are written as percentages of E_F: $T_M = 78.1\%$, $T_s = 9.8\%$, $T_L = 11.0\%$, $V_L = -9.0\%$, $V_s = 10.9\%$. These percentages total 100.8% as a consequence of accumulative errors in the numerical integration. Note that the limb potential energy is negative and that the total potential energy in the bow limbs and string system is only 1.9% at arrow exit.

Let us consider small vibrations for the elastic string case. We will find two fundamental frequencies corresponding to the two degrees of freedom, x and s, for the elastic string case, contrary to the single frequency

$$\omega_0 = \sqrt{3\kappa/m_s}$$

obtained for the inelastic string. We recall that for the latter case we found $f_0 = 82$ Hz when the nonlinear terms in the equation of motion were neglected and $f = 65.5$ Hz when those terms were taken into account. For the elastic string we will consider only the linearized equations of motion. This results in the following two equations of motion:

$$\ddot{x} + (1/2)\ddot{s}\cot\theta_0 + \omega_0^2(x - x_B) - \omega_0^2(s - s_B)\cot\theta_0 = 0,$$

$$\frac{1}{2}\ddot{x}\cot\theta_0 + \left(1 + \frac{6m_L\beta}{m_s}\right)\ddot{s}\csc^2\theta_0 - \omega_0^2(x - x_B)\cot\theta_0 + \omega_0^2\left(\frac{4k}{\kappa} + \cot^2\theta_0 + \frac{s_0}{l}\frac{(\sin\theta_0 - \theta_0\cos\theta_0)}{\theta_0\sin^2\theta_0}\right)(s - s_B) = 0.$$

Substitutions of an $e^{i\omega t}$ time dependence gives the usual secular determinant for ω with the following roots:

$$\omega = 0.670\,\omega_0 \quad\text{and}\quad 1.066\,\omega_0.$$

The two frequencies for the elastic string case are then

$$f = 55\text{ Hz} \quad\text{and}\quad 87\text{ Hz}.$$

VI. AIR RESISTANCE AND HYSTERESIS ENERGY LOSSES

The analyses of the dynamics of the bow and arrow in Secs. IV and V ignored any possible effects of air resistance or hysteresis losses. Estimates of air resistance effects have been made, as briefly summarized in Appendix B, and indicate the total energy dissipated by the limbs, string, and arrow to be 1.4%, or less than 2% of the total energy stored at full draw. This is small compared with either the 9% or 22% values of energy not available for kinetic energy of the arrow as predicted by the inextensible and elastic string models, respectively.

No hysteresis data exists that is of sufficient quality to clearly eliminate this effect as an energy-loss mechanism. However, the absence of data itself lends credence to its unimportance (it is difficult to make 1% force-draw curve measurements to determine the static hysteresis and the dynamical measurement would be more difficult yet). Crude results indicate the static hysteresis losses are less than 3% for contemporary bows of quality design.

VII. CONCLUSION

A comparison of the efficiency calculated by the inelastic model with the efficiency predicted by the elastic model shows a significant difference. The weight of experimental evidence, which has not been exposed in this paper, favors the elastic model. A few efficiency measurements are reported[3] in *Archery: The Technical Side.* If a mental integration of the results presented there is performed, the implications are that the half-round cross-sectional area long bows tested by Hickman, circa 1930, had efficiencies on the order of 60%. The rectangular cross-section bows tested by Klopsteg in the next decade fell into a 70% efficiency category. In the popular literature of the next two decades efficiency measurements were occasionally reported and fall in the 60–85% efficiency range. A few efficiency values over 90% were reported, but rarely, and never with sufficient explanation of the measurement technique to verify the reported value, nor assess the experimental errors. Measurements of efficiency made by the author, circa 1960, on some of the more popular bows of the time can be summarized as 80%. The totality of these unpublished efficiency measurements is in substantial agreement with the elastic string model and does not support the inelastic string model predictions. The latter models usually rely on air resistance and hysteresis losses to explain away the discrepancy between the theoretical predictions and experimental measurements; but as has been shown quantitatively, air resistance can account for only less than 2% energy loss and hysteresis for less than 3%. Dissipation is therefore a highly improbable explanation of the discrepancy.

In Hickman's first published article[1] he made measurements to obtain the time dependence of the arrow position, velocity, and acceleration. Except for the zero value of acceleration obtained by Hickman at arrow release (which was probably a consequence of his measurement technique and graphical differentiation to obtain velocity and acceleration), these acceleration profiles show the same undulations as result from the elastic string model for the acceleration temporal profiles. No comparison of the data with experiment can be made in detail, however, since not enough of the physical characteristic parameters required by the prediction model are available. Moreover, the inextensible string models show similar variations in the acceleration profiles. However, the elastic string model appears to be of adequate accuracy to justify remeasurements of temporal profiles of arrow position, velocity, and acceleration of contemporary bows.

The string tension at arrow exit as predicted by the elastic model is roughly half of the value predicted by the inelastic model. Unfortunately, no experimental data exists for string tension at arrow exit. However, Nagler[3] reports that for long life the breaking strength of the string should be four times the full draw force. Let us assume this rule of thumb. Then a safety factor can be defined as $4F_F/P_A$ for long life of the string. Using this prescription, the safety factors derived from the elastic and inelastic models, respectively, are 1.9 elastic and 1.1 inelastic. A safety factor of 1.1 seems very marginal. As a second point of substantiation of the elastic model, Taylor[3] indicates that he has found (without expounding on how) that the dynamic string tension at arrow exit is generally a factor of 2 greater than the static force at full draw. This factor is 2.1 for the elastic model and 3.7 for the inelastic model, again favoring the elastic model.

The static force required to hold the bow at full draw and the dynamic force on the arrow just after arrow release are not the same for either the elastic or inelastic models. The ratios of these two quantities are given in Table III for both models and there is nearly a 45% difference between them. The fundamental difference between the two models is that, for the elastic model, the static string tension at full draw is $2k(s_F - s_0)$ and the dynamic string tension at arrow release must also be $2k(s_F - s_0)$, since $\dot{s}_F = 0$. For the inelastic model the string tension is unrelated to the length of the string and becomes whatever value is needed in the force balance required by the system.

The analysis presented in this paper assumed infinite string signal velocities for motion both along the string and perpendicular to the string. The longitudinal signal velocity is

$$C_\parallel = \sqrt{4ks_0^2/m_s} = 1700 \text{ m/sec},$$

so that the time required for a signal to transit the string halflength is $t_\parallel = s_0/C_\parallel = 0.5$ msec. The transverse signal velocity is

$$C_\perp = \sqrt{2Ps_0/m_s} = 220 \text{ m/sec}$$

for string tension on the order of 200 N. The time required for a transverse signal to transit the string halflength is then $t_\perp = s_0/C_\perp = 3.9$ msec. Both of these times are greater than the step sizes used in the numerical integration and the transverse signal transit time is a significant fraction of the 16 msec required for the arrow to exit the bow. Consequently, the approximation used for the dynamics of the string should be reconsidered. This, unfortunately, would add considerable complexity to the model.

The model used for the estimate of air resistance on the motion of the string and bow limbs is admittedly crude and a more accurate approach is warranted. However, this will probably only be stimulated by more accurate experimental data. An interesting and not difficult experiment could be performed that might considerably enhance our understanding of the effects of dissipation within the string and bow limb system. This would be the measurement of the frequency and amplitude content of a plucked bow string. There is a significant difference between the predicted vibrational frequencies of the elastic and inelastic models and this might be used to determine their range of validity.

Aside from the obvious use of the elastic string model to carry out parametric studies such as determining the variation of efficiency with string mass, arrow mass, and limb mass or the variation of arrow velocity with brace height, bow limb length, riser length, etc., it might prove interesting to extend the model to include the effect of asymmetry about the x axis. When the bow and string system is no longer symmetric about the arrow, the system might generate an angular momentum, causing the bow to rotate about an axis normal to the plane of the bow limbs and string. Indeed, the modern archer has attached stabilizers of high moment of inertia to the nonworking portions of his bow to counteract torques he experiences in loosing his arrow.

The above comments indicate that the most useful future activity for advancing our understanding of bow and arrow dynamics would be the collection of accurate data for the dynamic arrow, string, and bow limb system with which the elastic string model can be compared.

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APPENDIX A: ARCHERY IN THE CLASSROOM AND LABORATORY

Many elements of archery may be used to exemplify physical principles in the classroom and laboratory. The bow and arrow have a universal familiarity that can be exploited in the classroom to hold student interest while he or she is absorbing such fundamental concepts as force, work, or potential energy. Many students will have had the opportunity to have drawn a bow and experience the force and work required to do so. Even those who may not would have little trouble in capturing the essence of a force-draw curve as related to a bow.

It is not a difficult transition from the force-draw curves such as those shown in Fig. 8 to the concept of potential energy and its equivalence to the work done in drawing the bow and to the energy stored under the force-draw curve. The construction of such a curve as a classroom demonstration can be accomplished inexpensively with a simple spring scale and meter stick. The force-draw curve construction requires little time and provides a graphic illustration of the principles involved. The energy stored under the curve is easily estimated and, certainly, the transfer of this stored energy to kinetic energy of the arrow is easy for the student to comprehend.

For the instructor or student with some archery ability the bow and arrow can be used in conjunction with the ballistic pendulum to measure the arrow velocity and consequently its kinetic energy. Coupling this result with the energy stored under the force-draw curve leads to the determination of the efficiency of the bow and arrow system. The ballistic pendulum measurment is difficult because of the angular momentum that can be imparted to the pendulum by off-center hits and because of the arrow's free vibration with amplitudes typically on the order of 2–3 cm. Adjustment of the distance from the point of arrow release to the pendulum on the order of a few arrow lengths will usually result in the determination of a distance at which the arrow strikes the pendulum parallel to its swing axis.

In introductory mechanics, one can ignore the mass of the string and bow limbs, neglect the elasticity of the string, and assume the force-draw curve to be linear. The bow and arrow dynamics then becomes very simple indeed. The energy equation is $(1/2)M\dot{x}^2 + (1/2)kx^2 = E$, where M is the arrow mass and the other parameters have their usual meaning. The static draw force is $F = kx$ and the equation of motion $M\ddot{x} = -kx$. The construction and understanding of these elementary equations of dynamics are easy for the student to grasp when the model is as simple and familiar as the bow and arrow. Moreover, the solution of these equations allows the introduction of the subleties of transient phenomena, a refreshing digression from the usual harmonic oscillator solution of these equations. The solution predicts reasonable values for the time for the arrow to transit the bow and for the final arrow velocity. A first approximation to reality is achieved with the recognition that the string does indeed have mass and must be moving at the time of arrow exit. A simple calculation shows that effectively 1/3 of the string mass moves with same velocity as the arrow at arrow exit, resulting in less than unity efficiency.

In intermediate mechanics one can give up massless strings and bow limbs to arrive at the inelastic string model. Without the complications of an elastic string, it is a fairly simple task to derive the energy equation for the bow and arrow system. This would give the student some practical experience with the dynamics of a variable mass system outside the realm of relativistic mechanics or the classical rocket problem. The depth to which the inelastic string dynamics is probed can vary from just the derivation of the energy equation and its use to determine approximately the final arrow velocity and bow and arrow system efficiency to the full expressions of the complete solution in closed form. The following exercise illustrates one manner in which the bow and arrow dynamics can be used. Assuming an inelastic string, linear force-draw curve, and a linear virtual mass-draw relation, what is the time for the arrow to reach the brace height position after arrow release? Under these assumptions the potential energy and virtual mass become $V = (1/2)k(x_B - x)^2$ and $K = K_F - (K_F - K_B)x/x_B$, respectively. The energy equation $(1/2)(M + K)\dot{x}^2 + V = E_F$ can then be integrated to give

$$t_B = \frac{2x_B}{\dot{x}_B}\left(\frac{M + K_F}{M + K_B}\right)^{1/2} E(\alpha,\pi/4),$$

where $E(\alpha,\phi)$ is the elliptic integral of the second kind and $\sin^2\alpha = 2(K_F - K_B)/(M + K_B)$. Other sample problems suited to the intermediate student level are What is the efficiency at brace height? and Compare the static force at full draw and the dynamic force on the arrow at the instant of release.

There are also opportunities for the intermediate student laboratory. The derivation of the small vibrations of a braced bow is a very simple matter when the equations are linearized for small vibrations. Similarly it should not be difficult or expensive to measure the predicted small displacement frequency in a normally equipped student laboratory (e.g., a small magnet taped to the string center, a loop of wire as a pickup coil, and an oscilloscope). Both nonlinearities and air resistances are important to the characterization of the free vibrations of a bow, in contrast to the case of the dynamics during arrow transit of the bow. The vibratory motion is damped in about 1 sec as the energy is dissipated to the air and possibly internally in the bow limbs. It is not difficult to predict damping times of this magnitude as due to air resistance, but it does require a familiarity with viscous aerodynamics outside of the simple Stoke's regime.

Finally, the full complexity of the elastic string problem

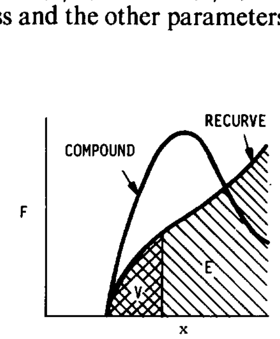

Fig. 8. Force-draw cruves for contemporary recurve and compound bows. The shaded area under the recurve force-draw curve labeled V is the potential energy for the draw position x. The total area under the force-draw curve is the total energy E stored in the bow at full draw.

could be introduced in a summary fashion in advanced mechanics classes. Just as in the ease of the introduction of forces, potentials, etc., for the beginning student, the bow and arrow system provides an interesting and easily understood framework for the introduction of the concept of Lagrangian densities. Moreover, the elastic string problem provides a practical example of a fairly simple derivation of a set of Euler–Lagrange equations in two degrees of freedom and an energy equation for a complex problem with variable mass that is readily interpretable in terms of familiar quantities. There is no known closed-form solution of these equations, but at this stage in a student's career he or she should be able to accept a numerical solution and understand the conclusions drawn from those results relative to the differences between the elastic and inelastic string models without a feeling of intimidation.

The transformation from the Euler–Lagrange equations of motion to the force diagrams for the bow and arrow system as shown in Figs. 4 and 5 of Sec. III is probably too tedious and lengthy for a classroom discussion. There is here, however, the genesis of a technique for teaching physics students the elements of complex force and torque systems. The method of generation of the Euler–Lagrange equations of motions of a complex system is a fairly straightforward problem for the advanced student, but the transcription of these equations to an understanding of the elementary forces and torques operating in a dynamic system and the possibility of treating these dynamic properties in much the same manner as static force and torque diagrams is foreign to the physic student. The approach used in this paper on bow and arrow dynamics using the Euler–Lagrange equations as the starting point seems to be a method that would be easily accepted by students with a typical physics curriculum background. In passing, it is noted that the bow and arrow dynamics affords one of the few practical examples where a Coriolis force exerts itself in a simple mechanical system. Typically, however, this Coriolis force is small relative to the other forces involved in the problem.

In summary, bow and arrow dynamics provides the introductory, intermediate, or advanced mechanics instructor with an interesting and familiar subject that can be utilized to introduce concepts often difficult for the student to absorb without such a concrete example.

APPENDIX B: AIR RESISTANCE EFFECTS

The effects of air resistance on the motion of the arrow, string, and bow limbs while the arrow transits the bow are considered in this appendix. Unfortunately, from a time just shortly after arrow release until the vibratory motion of the string is nearly complete, the velocities of the arrow, string, and limbs are too large to allow the simplifying approach of the Stokes approximation, i.e., a linear relationship between the retarding air resistance force and the velocity of motion.

Consider first the effect of air resistance on the arrow motion. Rheingans,[3] in a paper addressing the flight of an arrow subsequent to exit from the bow, developed the following empirical relation for the arrow drag force,

$$R_A = K\dot{x}^2,$$

where

$$K = K_1Bd^2 + K_2Dd + K_3A_F.$$

The parameter B provides an adjustment for the shape of the arrow head on tip, d is the diameter of the arrow, D the arrow length, and A_F the total area of the fletching. Fitting this expression to experimental data, Rheingans found

$$K_1 = 9.6 \times 10^{-2}\ \mathrm{N\,sec^2/m^4}\ (1.3 \times 10^{-6}\ \mathrm{lb\,sec^2/in.^2\,ft^2}),$$

$$K_2 = 2.6 \times 10^{-3}\ \mathrm{N\,sec^2/m^4}\ (3.5 \times 10^{-8}\ \mathrm{lb\,sec^2/in.^2\,ft^2}),$$

$$K_3 = 5.7 \times 10^{-3}\ \mathrm{N\,sec^2/m^4}\ (7.7 \times 10^{-8}\ \mathrm{lb\,sec^2/in.^2\,ft^2});$$

$$B = 1.0 \quad \text{for ogival tips,}$$

$$2.3 \quad \text{for parallel tips,}$$

$$7.0 \quad \text{for blunt tips.}$$

A reasonable approximation for the arrow velocity (in lieu of a numerical integration of the equation of motion) is given by, $\dot{x} = \dot{x}_A(2x_Ax - x^2)^{1/2}/x_A$. Then the energy loss due to air resistance as the arrow transits the bow is

$$\Delta E_A = \int_0^{x_A} R_A\,dz = \frac{2}{3}K\dot{x}_A^2x_A,$$

and the percentage of the total energy is

$$\Delta E_A/E_F = 0.1\%.$$

These calculations assumed an arrow with $B = 1$, $d = 0.79$ cm (20/64 in.), $D = 70$ cm (28 in.), and $A_F = 58\ \mathrm{cm^2}$ (9.0 in.^{2}).

Determination of the energy loss for the string and limbs is somewhat more involved. The string loss will be approximated by

$$\Delta E_s = \rho C_{D_s}d_s \int_0^{s_0} d\xi \int_{x_s(0)}^{x_s(x_A)} dx_s\,\dot{x}_s^2,$$

where ρ is the air mass density, C_{D_s} the drag coefficient, d_s the string diameter, $d\xi$ an elemental length of the string, and dx_s an infinitesimal displacement of the latter string element parallel to the x axis. Motion of the string in the y direction is neglected. Using previously given relations for x_s and $\dot{x}_s$, then

$$\Delta E_s = \rho C_{D_s}d_s \int_0^{s_0} d\xi \int_0^{x_A} dx\,\dot{x}^2(1 + \xi\varphi_x\cos\varphi)^3.$$

Approximating the expression in the brackets as a linear function of x between its end point values of 0.36 at $x = 0$ and 0.25 at $x = x_A$, then

$$\Delta E_s = 0.19\rho C_{D_s}d_ss_0\dot{x}_A^2x_A$$

and

$$\Delta E_s/E_F = 1.2\%,$$

for values of the parameters as follows: $\rho = 1.29 \times 10^{-3}$ g/cm^3, $C_{D_s} = 1.0$,[12] $d_s = 0.2$ cm, $s_0 = 85$ cm, $\dot{x}_A = 52$ m/sec, and $x_A = 0.46$ m.

The energy loss of the limbs is approximated by

$$\Delta E_L = \rho C_{DL} \int_0^l dq\,w \int_{r(0)}^{r(x_A)} dr\,\dot{r}^2,$$

where the notation follows that used previously. With $w = w_0 + w_1 - w_1q/l$ and $r = q\theta$, this becomes

$$\Delta E_L = \frac{1}{4}\rho C_{DL}lw_0\left(1 + \frac{1}{5}\frac{w_1}{w_0}\right) \int_0^{x_A} dx\,\dot{x}^2\,(l\theta_x)^3.$$

Assuming the function $l\theta_x$ to be proportional to x, then

$$\Delta E_L = 0.0018\rho C_{DL} l w_0 \left(1 + \frac{1}{5}\frac{w_1}{w_0}\right) \dot{x}_A^2 x_A$$

and for $C_{DL} = 1.6$,[13] $l = 0.48$ m, $w_0 = 1$ cm, and $w_1 = 1.2$ cm, then

$$\Delta E_L/E_F = 0.1\%.$$

The total energy dissipated by the limbs, string, and arrow is therefore 1.4% or less than 2% of the total system energy.

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[13]Reference 12, p. 3-17, Fig. 33(d).