The Dynamics of A Bow and Arrow

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A SERIES of articles by the author dealing with the static strains and stresses in a drawn bow have been published in *Ye Sylvan Archer*.[1] These articles have shown the effect of the shape and form of bending of the bow upon these strains and stresses. Some interesting and valuable information was obtained from that work which has materially changed the design of modem bows.

Considerable time has been spent by the author in studying, by experimental methods, the behavior of a bow while the arrow is being discharged. Some of the results of these investigations have been published in the magazine previously mentioned and in *The Journal of the Franklin Institute* (October, 1929). He has also had high speed pictures (one to three thousand frames per second) taken by Electric Research Products which, in addition to putting an end to all controversies regarding the "archers paradox" furnish an excellent means of studying the motions of the arrow and bow limbs.

While these experimental methods are interesting and valuable, they are necessarily slow and expensive. An analytical method was therefore developed for determining the dynamical forces, the velocities and accelerations of the arrow and bow limbs. This method has been useful in improving the design and cast of bows. The method is briefly set forth in this paper. Since the dynamical treatment depends on the static conditions, a few of the static equations will first be given.

Let us assume that the bow is made up of a short rigid middle section and two limbs, which, for all positions of the draw, bend in arcs of circles. This assumption is reasonable because practically all bowyers construct their bows so that they bend in this form. In the articles referred to above, it has been shown that a very simple design having this form of bending will stress all sections of the bow equally. It is believed that for most bows this form of bending is desirable. If for some particular bow another form of bending is desired, the method used here may be applied to it.

Referring to Fig.1, let B equal one-half the length of the bow, L equal one-half the length of the rigid handle section, $B_1 = B+L$ equal the length of the active bending portion of each limb, S equal one-half the length of the string, H equal the distance from middle of bow to middle of line connecting bow tips, P equal distance from arrow nock to middle of line connecting bow tips, $D = H+P$ equal length of draw, Y equal one-half length of line connecting bow tips, A equal angle between the line connecting bow tip and point Q and the line representing the position of the undeflected limb. (The point Q is located at a distance of $3B_1/4$ from the tip of the bow.) Let E equal angle made by string with the line connecting the bow tips, f equal the static force at each bow tip in a direction tangent to its path, T equal static tension in the string, F equal static drawing force, $Y_1 = Y–B_1/4–L$ and N equal the distance along the path made by the bow tip during the draw.

It may be shown that the path made by the bow tip is part of a cardioid. The portion of the cardioid traversed by the bow tip is almost a perfect arc of a circle whose radius is $3B_1/4$ and whose center is located at a distance of $3B_1/4$ from the tip of the undeflected bow. Since the equations which are to be obtained are based on the assumption that the bow tips travel along arcs

[1] Nov. 1930 to Aug. 1932 (published at Albany, Oregon)

of circles having radii of $3B_1/4$, the accuracy of this assumption will now be investigated.

In Fig.2, let B_1 represent the length of the bending portion of one of the bow limbs. Since the limb bends in the arc of a circle, B_1 is the arc of a circle having a radius of r and the angle θ corresponds to this arc. The chord $K = 2r\cdot sin\theta/2$ and $r = B_1/\theta$ where θ is measured in radians. Therefore $K = (2B_1/\theta)\cdot sin\theta/2$. The point Q is selected so that $QY = 3B_1/4$ and $QX = B_1/4$. $XY = B_1$ and represents the length and position of the active bending portion of the limb before the draw. The angle $QXZ = \theta/2$. Let $QZ = R$. Then:

$$R = [B_1^2/16 + K^2 - (KB_1/2)\cdot cos\theta/2]^{1/2}.$$

Substituting the above value of K and making use of well-known trigonometric relations it may be shown that:

$$R = (3B_1/4)\cdot(1 - 16\theta^4/9\,\underline{|6}+\ldots)^{1/2}.$$

Since the angle θ for a bent bow never exceeds unity (i.e. 57.3°), all the terms under the radical except the first one are negligible. Therefore $R = 3B_1/4$. (The maximum value of θ for the average bow is equal to about *I.* For this value of θ, the second term under the radical would equal 16/32,805. Neglecting this term results in an error of less than two parts in 1000.)

Referring again to Fig.1, the force f is proportional to the displacement N and. is therefore proportional to the angle A which corresponds to the arc N.

Let $f = CA$ where C is a constant depending on the dimensions and material of the bow.

A subscript O will be used to indicate the value of any of the above variables when the bow is in its braced but undrawn position.

The following trigonometric and algebraic relationships may now be obtained:

$Y_1 = (3B_1/4)\cdot cosA$, $\quad Y = Y_1+B_1/4+L$,

$S = Y_0 = (3B_1/4)\cdot cosA_0+B_1/4+L$

$= \frac{1}{2}$ length of string,

$H = (3B_1/4)\cdot sinA$, $\quad H_0 = (3B_1/4)\cdot sinA_0$,

$E = cos^{-1}Y/S$, $\quad P = S\cdot sinE = (S^2-Y^2)^{1/2}$,

$D = H+P$, $\quad T = f/sin(A+E) = CA/sin(A+E)$,

$T_0 = f/sinA_0 = CA_0/sinA_0$,

$F = 2T\cdot sinE = (2AC\cdot sinE)/sin(A+E)$,

$N = 3B_1A/4$, $\quad N_0 = 3B_1A_0/4$.

Dynamical Treatment

Let x represent the distance traveled by the arrow in t seconds. Then: $dx/dt = V$ where V is the instantaneous velocity of the arrow. But $x = D'-D$ where D' is the value of D for the fully drawn position. (Prime letters will be used to represent the value of the variables for the fully drawn position. Prime letters will therefore be constants.)

$$dx/dt = d(D'-D)/dt = -dD/dt = V .$$

In like manner: $-dN/dt = v$ equal velocity of bow tip. But: $dD/dN = (dD/dt)/(dN/dI) = V/v = R$ where R is the ratio of the two velocities.

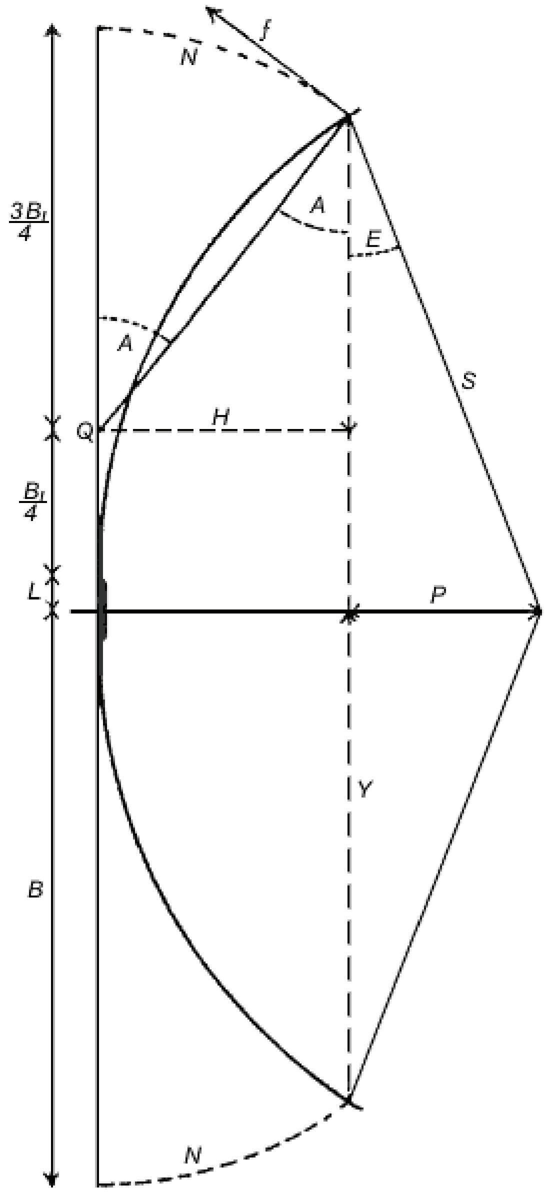


FIG. 1.

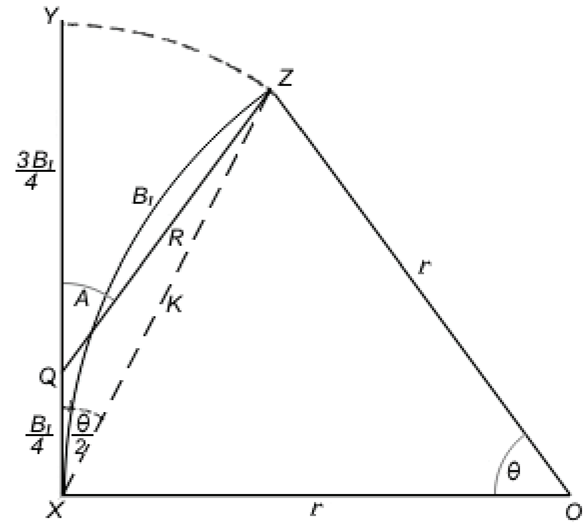


FIG. 2.

Also,

$$dD/dN = (dD/dA)(dA/dN)\ .$$

But $D = H+P$. Therefore, substituting the values given for $D = H+P$ and differentiating

$$dD/dA = (3B_1/4)\cdot cosA + (3B_1\ Y/4P)\cdot sinA\ .$$

But $A = 4N/3B_1$ and $dA/dN = 4/3B_1$. Therefore:

$$dD/dN = V/v = R = cosA+(Y/P)\cdot sinA\ .$$

Let M equal mass of arrow, $m/2$ equal effective mass of each limb located at tip and W equal potential energy of both limbs. Since N is the displacement of each bow tip: $W'–W = C(A'N'–AN')$ is the loss in potential energy of both limbs from position of full draw to any other position.

Neglecting the dissipation in heat and sound, the gain in kinetic energy is equal to the loss in potential energy. Therefore:

$$MV^2/2 + mv^2/2 = C(A'N'–AN)\ .$$

But $V = Rv$. Therefore:

$$v = [2C(A'N'– AN)/(MR^2+m)]^{½}\ .$$

But $N = 3B_1A/4$. Therefore:

$$v = [3B_1C(A'^2–A^2)/2(MR^2+m)]^{½}\ .$$

And

$$V = Rv = R(3B_1C\cdot)(A'^2–A^2)/2(MR^2+m)^{½}\ .$$

Let **a** equal the acceleration of the bow tip and **A** the acceleration of the arrow. (Bold face type will be used to indicate dynamic values of variables.)

$$\mathbf{a} = dv/dt = (dv/dA)(dA/dt)$$
$$= (dv/dA)(dA/dN)(dN/dt).$$

But dA/dN = 4/3B1 and dN/dt = –v. Therefore: $\mathbf{a} = –\ (4v/3B_1)(dv/dA)$. Therefore by differentiation:

$$\mathbf{a} = 2CA/(MR^2+m)$$
$$– 2MRC(A'^2–A^2)Z/(MR^2+ m)^2\ .$$

where

$$Z = sinA–(Y/P)cosA+(3B_1S^2/4P^2)sin^2A\ .$$

Also $\mathbf{A} = dV/dt = d(Rv)/dt = vdR/dt+Rdv/dt$.

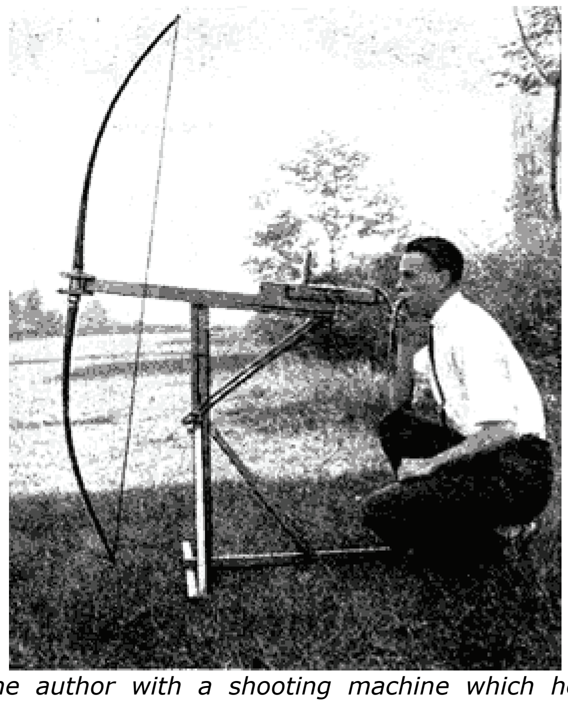
$$dR/dt = (dR/dA)(dA/dt)$$
$$= (dR/dA)(dA/dN)(dN/dt) = –(4v/3B_1)dR/dt\ .$$

Therefore

$$\mathbf{A} = 2CAR/(MR^2+m)$$
$$+2mC(A'^2–A^2)Z/(MR^2+m)^2\ .$$

The dynamical force **f** on the tip is: $\mathbf{f} = m\mathbf{a}/2$.
The dynamical force **F** on the arrow is:

$$\mathbf{F} = M\mathbf{A}.$$



The author with a shooting machine which he designed for use in testing arrows. This machine was used in conjunction with a chronograph for measuring the velocity and acceleration of arrows. It is equipped with pneumatic release.

The dynamical tension in the string is:

$$\mathbf{T} = \mathbf{F}/2\cdot sinE = S\mathbf{F}/2P = SM\mathbf{A}/2P = (f – \mathbf{f})/sin(A+E)$$

If the angle A is expressed in radians, the masses in pounds, the lengths in feet and the forces in poundals, the velocity and acceleration will be obtained in feet per second and feet per second per second, respectively.

It is more convenient, however, to use degrees for the angle A, inches for the lengths, pounds for the forces and grains for the masses. It is also desirable to have the constant C of such a value that $f = CA$ where f is expressed in pounds and A is expressed in degrees. In order to use the units as just explained and still obtain the velocities and accelerations in feet per second and feet per second per second, respectively, the above formulas become:

$$\mathbf{v} = [492B_1C(A'^2 – A^2)/(MR^2+m)]^{½}, \qquad \mathrm{V} = R\mathbf{v}$$

$$\mathbf{a} = 450{,}800\cdot CA/(MR^2+m) – 7{,}868\cdot MRC(A'^2 – A^2)Z/(MR^2+m)^2$$

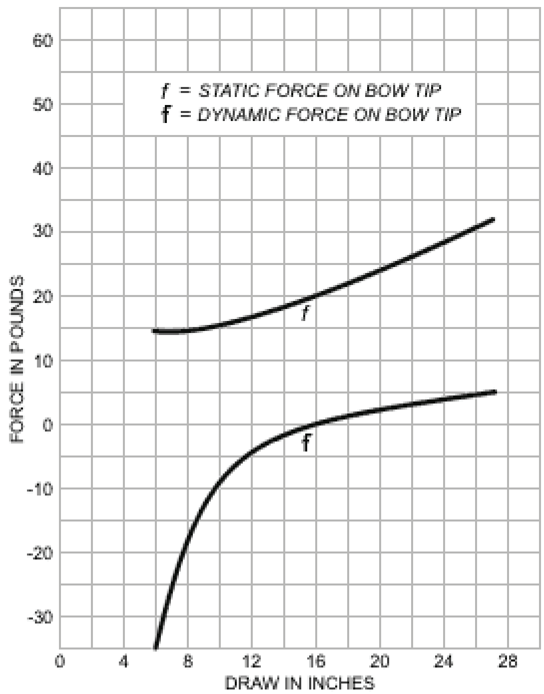


FIG.3

$$A = 450{,}800\cdot CAR/(MR^2+m) + 7{,}868\cdot m\, C(A'^2 – A^2)Z/(MR^2+m)^2,$$

$$\mathbf{f} = 2.22\cdot ma\cdot 10^{-6}, \qquad \mathbf{F} = 4.44\cdot MA\cdot 10^{-6},$$

$$\mathbf{T} = \mathbf{F}/2\cdot sinE = S\mathbf{F}/2P = 2.22\cdot SM\mathbf{A}\cdot 10^{-6}/P = (f – \mathbf{f})/sin(A+E)$$

$$R = \mathbf{V}/\mathbf{v} = cosA + (Y/P)\cdot sinA$$

In order to show how these formulas may be used, they are now applied to a six-foot bow which has an eight-inch rigid middle section (L = 4 inches), and which has a six-inch bracing height (H_0 = 6 inches), and a draw D = 27.5 inches.

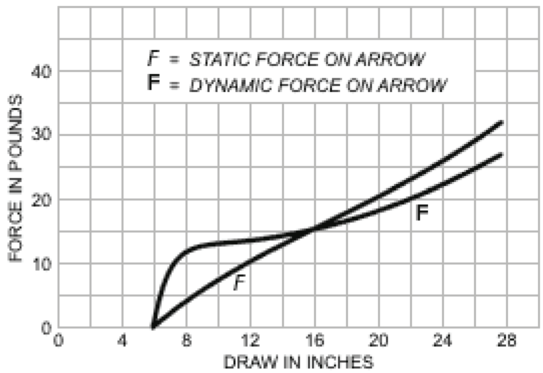


FIG.4

From the equation $H_0 = (3B_1/4)\, sinA_0$,

$$A_0 = 14° 29'$$

From the equation $S = (3B_1/4)\cdot cosA_0 + B_1/4 + L$,

$$S = 35.2373 \text{ inches.}$$

The active bending portion of each limb,

$$B_1 = 32 \text{ inches.}$$

Let each limb have a constant thickness, t = 0.6 inch. Let the width at the dip equal W = 1.5 inches. Let the width at any other point equal $w = Wx/B_1$, where x is the distance from the nock. This type bow is discussed in the January, 1932 issue of *Ye Sylvan Archer.* All sections of such a bow are equally stressed and the limbs bend in true arcs of circles.

It may be shown that the kinetic energy of a limb for this type bow $\mathbf{K} = tWdB_1\mathbf{V}^2/60$ where d is the density of the bow wood and $\mathbf{V}$ is the velocity of the nock as it passes through its neutral position.

If a bow limb having the same kinetic energy has all its mass $m/2$ concentrated at the nock,

then: $mV^2/4 = tWdB_1V^2/60$

and $m = tWdB_1/15$.

For m in grains, t, W, and B_1 in inches and d in pounds per cubic foot, this equation becomes:

$$m = 0.270\cdot tWdB_1.$$

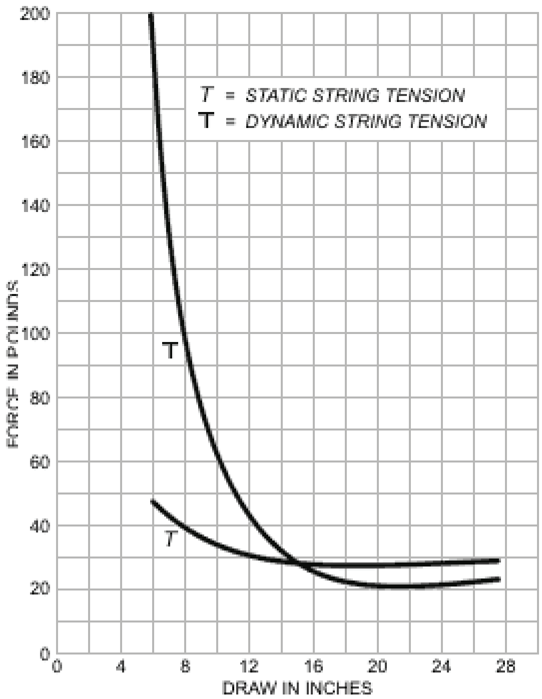


FIG.5

For yew wood, $d = 43$. Therefore:

$$m = 0.270 \cdot 0.6 \cdot 1.5 \cdot 43 \cdot 32 = 334.4 \text{ grains.}$$

If the arrow weighs 350 grains and the string weighs 105 grains, M will equal 385 grains, because it has been shown in previous publications that approximately 1/3 the weight of the string should be added to the weight of the arrow in order to obtain its effective mass M.

It is now necessary to determine the value of the constant C.

In the January, 1932 issue of *Ye Sylvan Archer*, it was shown that the deflection at the nock of this type bow $N = 6fB_1^2/Wt^2Y$ where Y is Young's modulus for bending. But $N = B_1\pi A/240$ where A is expressed in degrees and

$$f = CA\ .$$

Therefore: $B_1\pi A/240 = 6CAB_1^2/W\,t^2\,Y$

from which $C = W\,t^2\,\pi Y/1440\,B_1$

For yew, $Y = 1.46 \cdot 10^6$.

Therefore $C = 1.005$.

The formulas developed in this paper are now applied and the results plotted in graph form.

Fig.3 shows the static and dynamic forces on the bow tip or nock, as a function of the draw.

The curve marked f represents the force tangent to the path required to deflect the tip for all positions of draw. When the string is released with the arrow in place, most of the potential energy of the bow is used in accelerating the arrow so that there is initially only about five pounds of force effective in accelerating the bow tip. This dynamical force f becomes smaller as the arrow is being discharged. It reaches zero for a position corresponding to a draw of about 15.5 inches. This force then becomes negative. This means that the static force tending to move the tips forward is not as great as the retarding force resulting from the string tension. This difference is made up by the kinetic energy that has been acquired by the tips. The force which is tending to stop the tips is now being used to accelerate the arrow.

Fig.4 shows the static and dynamic forces on the arrow. Here we see that at the moment of loose, the dynamic force **F** on the arrow is smaller than the static force F which was on

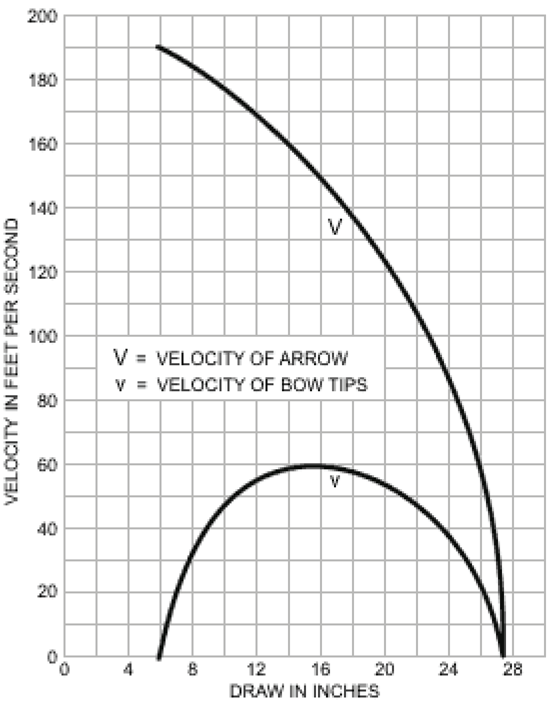


FIG.6

the holding fingers. This is due to the fact that some force is required to start the limbs in motion. As the arrow moves forward, the force on it decreases as shown by the curve **F**. The dynamical force **F** does not decrease as rapidly as the static force *F.* This is due to the kinetic energy of the tips. This explains why massive bow tips do not reduce the velocity of the arrow by a very great amount. The energy required to put the tips in motion is given to the arrow at a later time. However, this increase in the force **F** due to the kinetic energy of the limbs may not be desirable from the standpoint of accuracy.

The static string tension *T* is given in Fig.5. In the same graph we find that as the arrow is loosed, the dynamical tension **T** drops slightly but soon begins to increase. Were it not for the fact. that the string may stretch slightly, this tension would break the string. Heavy tips increase this tension.

Fig.6 gives the velocity of the arrow and the velocity of the bow tips as a function of the draw. We see that the velocity of the bow tips becomes a maximum corresponding to a draw of about 15 inches. It then becomes smaller and smaller, reaching zero at the time the string reaches its neutral position. Because of stretching of the string this zero velocity is reached slightly later than shown. However, in this case the tips actually reverse their direction and move back again.

A heavy string accentuates this effect. The velocity of the arrow increases throughout the time of discharge. However, for light arrows and heavy tips, the rate of increase may become much less during the middle portion of the discharge and then increase quite sharply near the end,

Fig.7 shows the accelerations of the arrow and of the bow tips as functions of the draw. The acceleration of the bow tips becomes smaller and smaller until it reaches zero at a position corresponding to a draw of about 15.5 inches. The acceleration then becomes negative. This means that the tips are slowing up. The acceleration of the arrow decreases for a short distance and then the decrease becomes less rapid. For very light arrows it may increase again. The shorter the bow, the more the reflex, the heavier the tips and the lower the stringing height, the more pronounced is this effect. That is why so many archers condemn all these conditions where accuracy is desired. A bow having a permanent set will reduce this effect. Heavy arrows will also prevent the acceleration from increasing near the close of the discharge. A whip-ended bow will also reduce the acceleration near the end of the discharge. This explains why a whip-ended bow does not kick.

The computations given here are for bow having an efficiency of 100 percent. If the string

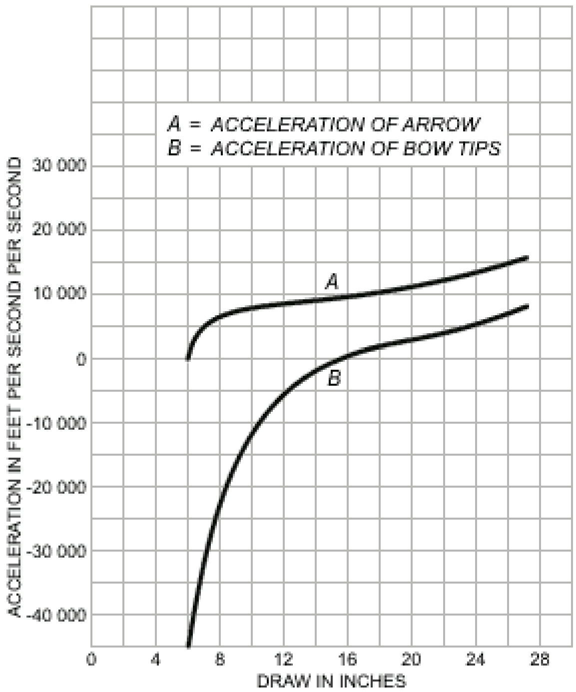


FIG.7

is taken into consideration as in the above computations, bows may be built that have an efficiency of 93 percent. An Osage bow of the type described above was built by Dr. P. E. Klopsteg which, after taking the string into consideration, had an efficiency of 93 percent. Most bows have much lower efficiencies. If the bow wood is good and no part is stressed beyond its elastic limit, the efficiency will be lowered only by the fact that the kinetic energy of the limbs is not all transferred to the arrow. The loss within the wood and in air resistance is usually negligible. If, in making the computations, the final velocity of the arrow is taken for a position about 2 inches in advance of the bracing height, a value will be obtained that compares well with experimental results. In other words, in practice all of the kinetic energy of the tips is not converted into arrow velocity as assumed in the computations.